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NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

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Asymptotic Methods for the Solution of Dispersive Hyperbolic Equations

ROBERT M. LEWIS

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Prepared for
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
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BEDFORD, MASSACHUSETTS

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ASYMPTOTIC METHODS FOR THE SOLUTION OF DISPERSIVE HYPERBOLIC EQUATIONS

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Robert M. Lewis

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ABSTRACT

A general method is presented for finding asymptotic solutions of initial-boundary value problems for linear hyperbolic partial differential equations. A large parameter λ appears in the equation, multiplying a lower order (dispersive) term. It also may appear in the initial data and the inhomogeneous (source) term, in a variety of ways. This gives rise to a variety of different types of asymptotic solutions. The expansion procedure is a "ray method", i.e., all of the functions that appear in the expansion satisfy ordinary differential equations along certain space-time curves called rays. These rays do not lie on characteristic hypersurfaces, but instead fill out the interior of the characteristic hypercone. They are associated with an appropriately defined group velocity. The details of the expansion are presented for a simple second order hyperbolic equation. The applicability of the method to symmetric hyperbolic systems and to the integro-differential equations for dispersive dielectrics is discussed.

1. Introduction

In recent years asymptotic methods have been developed for the solution of certain boundary-value problems and initial-value problems for partial differential equations. These problems involve a parameter, and the methods provide one or more terms of the asymptotic expansion, say for large values of the parameter, of the solution of the problem. They are often applicable to problems for which exact solutions are not known, and even for problems which can be solved exactly it frequently happens that only the asymptotic expansion of the solution is sufficiently simple to be useful in practical applications. Furthermore, it is invariably true that the methods which yield the asymptotic expansion directly are very much simpler than the procedure which involves first finding the exact solution, and then its asymptotic expansion.

An important class of asymptotic methods is characterized by the fact that certain curves, often called "rays", play a central role in the theory. The rays are of fundamental importance because all of the functions which make up the various terms of the expansion can be shown to satisfy ordinary differential equations along these curves. Thus, in a sense, the method is one which reduces partial differential equations to ordinary differential equations.

The "ray method" has been extensively developed, primarily by J. B. Keller and his co-workers at New York University, for the reduced wave equation

$$\Delta v + k^2 n^2(X) v = 0 \tag{1}$$

and related elliptic equations. When applied to (1) the method yields a "geometrical theory of diffraction"[1] which generalizes the classical theory of geometrical optics. The extensive literature on the asymptotic theory of the reduced wave equation and Maxwell's equations has recently been unified and summarized in [2]. The reduced wave equation is, of course, obtained from the wave equation

$$\Delta u - \frac{1}{c^2(X)} u_{tt} = 0 \tag{2}$$

by the substitution $u(t,X) = v(X)e^{-i\omega t}$. Then $k = \omega/c_o$ and $n(X) = c_o/c(X)$. (c_o is a convenient constant. The large parameter is ω or k.) Similarly the "symmetric hyperbolic" system of equations

$$\sum_{\nu=0}^{n} A^{\nu} \frac{\partial u}{\partial x_{\nu}} + Bu = 0\,, \quad (x_o = t), \tag{3}$$

is reduced by the substitution $u(t,X) = v(X)e^{-i\omega t}$ to the system

$$\sum_{\nu=1}^{n} A^{\nu} \frac{\partial v}{\partial x_{\nu}} + (B - i\omega)v = 0\,. \tag{4}$$

Here $X = (x_1, \ldots, x_n)$, A^{ν} and B are real $k \times k$ matrices which are functions of X, A^o is positive definite, and $A^1, \ldots, A^n$ are symmetric. The asymptotic theory of (4) for large ω has been studied in [3].

The original hyperbolic system (3) contains no parameter. However, P. D. Lax [4] has obtained the asymptotic solution of the initial-value problem for (3) with "oscillatory initial data" of the form

$$u(0,X) = e^{i\lambda s(X)} z(X). \tag{5}$$

In Lax's problem the large parameter λ appears only in the initial data, and specifically in the form (5). The role of the rays is played by the "bicharacteristics" which are space-time curves that generate the characteristic hypersurfaces of (3).

In this paper we shall examine a method which is applicable to the solution of initial value and initial-boundary value problems for symmetric hyperbolic systems of the form

$$\sum_{\nu=0}^{n} A^{\nu} \frac{\partial u}{\partial x_{\nu}} + \lambda\, Bu = f(t,X;x) \tag{6}$$

with initial conditions

$$u(0, X) = u_{o}(X; \lambda). \tag{7}$$

The large parameter λ appears in the system of differential equations (6), multiplying the undifferentiated term Bu. It is well known (See [5], p. 191) that the presence of such terms leads to the phenomenon of dispersion*, and the large paremeter in the dispersive term ensures that these phenomena appear in the asymptotic solution. For this reason the asymptotic method when applied to (6) leads to results which are quite different from the solution of Lax's problem. For the dispersive system (6) the bicharacteristics no longer serve as "rays." In fact the rays do not lie on characteristic hypersurfaces, and they represent trajectories of points which move with an appropriately defined "group velocity." Thus, for example, the asymptotic construction of the Riemann function, which is defined by (6) and (7) with $f = 0$ and $u_{o} = \delta(X)$, leads to rays that emanate from the origin in space-time and fill out the interior of the characteristic conoid. The parameter λ also appears in the source term f of (6) and the initial data

---

[*] For dispersive systems, different frequency components of a wave travel with different velocities; this leads to distortion of wave forms and other interesting physical phenomena.

function u_0 of (7). We will see that the simplest type of expansion is obtained when u_0 is oscillatory, i.e., of the form (5). However it is possible to obtain asymptotic solutions of (6,7) when λ appears in u_0 and f in a variety of different ways. This leads to a variety of different types of asymptotic solution, each of which is characterized geometrically by a typical ray pattern. Some of these will be examined in sections 3 and 4 of this paper.

An interesting example of a dispersive symmetric hyperbolic system of the form (6) is given by

$$\nabla \times H - \epsilon \frac{\partial E}{\partial t} = J, \quad \nabla \times E + \mu \frac{\partial H}{\partial t} = 0, \quad \frac{\partial J}{\partial t} - \epsilon \phi^2 E = 0. \qquad (8).$$

The first two equations are simply Maxwell's equations for the electric and magnetic fields E, H in M.K.S. units. The third equation is obtained by assuming that the current J is entirely due to the motion of electrons with charge e, mass m and number density $\eta(X)$, and that these electrons satisfy Newton's equation with the force term given by eE. In (8) ϕ is the "plasma frequency" defined by

$$\epsilon\phi^2 = \frac{\eta e^2}{m}. \qquad (9)$$

The system (8) describes the propagation of electromagnetic waves in certain plasmas such as the ionosphere. For the case of harmonic time dependence (8) reduces to the well known equations of the (isotropic) magneto-ionic theory[7]. Let us introduce the velocity of light $c = (\epsilon\mu)^{-\frac{1}{2}}$, an average plasma frequency ϕ_0, and a characteristic length* a. We then replace E,H,J,X,t in (8) by E', H', J', X', t' and set

$$\sqrt{\epsilon}\, E' = E, \sqrt{\mu}\, H' = H, J' = \sqrt{\epsilon}\, \phi J, X' = aX, ct' = at, \phi(X) = \phi_0 b(X). \qquad (10)$$

---

*For problems with boundaries a denotes a typical boundary dimension. For problems with non-constant $\phi(X)$ a is an average value of the function $\frac{\phi(X)}{|\nabla\phi(X)|}$. Thus $\phi(X)$ changes by a small fraction of itself over distances small compared to a.

Then (8) becomes

$$E_t - \nabla \times H + \lambda b(X)J = 0, \quad H_t + \nabla \times E = 0, \quad J_t - \lambda b(X)E = 0. \tag{11}$$

The (large) parameter λ is given by

$$\lambda = \frac{\phi_o a}{c} \,. \tag{12}$$

X and t are dimensionless space and time variables and b(X) is a dimensionless function proportional to the plasma frequency.

It is now an easy matter to verify that (11) is of the form (6) with $f = 0$. In this case u is the column vector with components

$$(E_1, E_2, E_3, H_1, H_2, H_3, J_1, J_2, J_3), \tag{13}$$

A^o is the 9×9 identity matrix, and A^1, A^2, A^3 are 9×9 matrices defined by

$$\sum_{\nu=1}^{3} k_\nu A^\nu = \begin{bmatrix} 0 & -(K) & 0 \\ (K) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \qquad B = \begin{bmatrix} 0 & 0 & bI_3 \\ 0 & 0 & 0 \\ -bI_3 & 0 & 0 \end{bmatrix} \tag{14}$$

Here I_3 is the 3×3 identity matrix and (K) is the 3×3 matrix

$$(\kappa) = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} . \tag{15}$$

Since (K) is anti-symetric we note that A^ν is symmetric and B is antisymmetric. The last condition implies that the system conserves energy and is important for our expansion procedure as we will soon see.

H and J are easily eliminated from (11) to obtain the second order system of equations

$$E_{tt} - \Delta E + \nabla(\nabla \cdot E) + \lambda^2 b^2(X)E = 0. \qquad (16)$$

If b is independent of x_3 and $E = \mu(x_1,x_2)A_3$, where μ is a scalar function and A_3 is a unit vector in the direction of the 3 axis, then $\nabla \cdot E = 0$ and (16) reduces to

$$u_{tt} - c^2(X)\Delta u + \lambda^2 b^2(X)u = 0, \qquad (17)$$

with $c = 1$.

Equation (17) is a simple example of a dispersive hyperbolic equation that conserves energy, and it serves admirably as a model for demonstrating our method of asymptotic solution. In Section (2),(3), and (4) we shall develop methods of obtaining asymptotic solutions of initial and initial-boundary value problems for (17) and for the same equation with an inhomogeneous (source) term. In our work we shall deal only with (17) and in sections (3) and (4) we shall, for simplicity, further restrict our considerations to the case of one space dimension. However, we emphasize that the methods are quite general. Research currently in progress has established that they are applicable to symmetric hyperbolic systems of the form (6) (with anti-symmetric B) in any number of dimensions. The main condition necessary for application of the method can best be described in terms of the "dispersion equation"

$$\delta(k_1, \ldots, k_n, \omega) = 0 \qquad (18)$$

which can be obtained by requiring that a plane wave of the form

$$u = \text{const. } \exp\left\{ i\lambda\left(\sum_{\nu=1}^{n} k_\nu x_\nu - \omega t \right) \right\} \tag{19}$$

be a solution of the partial differential equation with constant coefficients. Then the condition is that the solutions

$$\omega = h(k_1, \ldots, k_n) \tag{20}$$

of (18) be real for real $k_1, \ldots, k_n$. (To what extent this requirement can be relaxed is not yet clear.)

For (17)

$$\omega = \pm\sqrt{c^2k^2+b^2} \quad ; \; k^2 = \sum_{\nu} k_\nu^2 \; ; \tag{21}$$

and for (6) (with constant coefficients and f = 0) the dispersion equation takes the form

$$\det G = 0; \quad G = Q - A^0 \omega \; ; \; Q = \sum_{\nu=1}^{n} k_\nu A^\nu - iB. \tag{22}$$

That this equation has only real solutions ω for real k_ν follows from the fact that A^0 is positive definite and Q is hermitean, for ω is an eigenvalue of the hermitean matrix $(A^0)^{-1/2} Q (A^0)^{-1/2}$.

Before proceeding to the detailed discussion of our method we would like to point out that it is also applicable to the very important case of dispersion of electromagnetic waves in dielectrics. This case is not included in (6) because the electric and magnetic field vectors E, H in a dispersive dielectric satisfy the system of differential and integral equations*

$$\nabla \times E + \frac{1}{c}\frac{\partial H}{\partial t} = 0, \quad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = 0,$$

$$D(t,X) = E(t,X) + \int_0^\infty f(\tau,X)E(t-\tau,X)d\tau. \tag{23}$$

Here we have, for simplicity, taken the magnetic permeability to be $\mu = 1$. The dielectric permeability ϵ is a function of frequency, and f is given by

$$f(t,X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}[\epsilon(\omega,X)-1]d\omega. \tag{24}$$

Since ϵ is dimensionless it must be a function of ω/ω_0 where ω_0 is a characteristic frequency of the dielectric. We now replace

$$X, t, \omega, \epsilon, f \quad \text{by} \quad X', t', \omega', \epsilon', f'$$

and introduce a characteristic length a. Then we set

$$X' = aX, \; ct' = at, \; \omega' = \omega_0\omega, \; \epsilon'(\omega', X') = \epsilon(\omega, X), \; af'(t',X') = cf(t,X). \tag{25}$$

With this transformation (23) and (24) become

$$\nabla\times E + \frac{\partial H}{\partial t} = 0 \quad , \quad \nabla\times H - \frac{\partial D}{\partial t} = 0, \quad D(t,X) = E(t,X) + \int_0^\infty f(\tau,X)E(t-\tau,X)d\tau, \tag{26}$$

$$f(t,X) = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\lambda t}[\epsilon(\omega,X)-1]d\omega. \tag{27}$$

Here

$$\lambda = \frac{\omega_0 a}{c} \quad . \tag{28}$$

---

* See [8], section 58.

Although the system of equations (26) is quite different from (6) a very similar asymptotic method is applicable to the solution of (26) for large λ.*

Let us indicate briefly the steps involved in the asymptotic method. For problems which can be solved exactly**, examination of the asymptotic expansion of the solution shows that it consists of a sum of terms, each of which is an asymptotic power series in λ^{-1} involving a "phase function" and an infinite sequence of "amplitude functions". For complex problems, we therefore assume that the solution is also given by a sum of such series. By inserting such a series into the partial differential equation, we find first that the phase function satisfies a first order partial differential equation which can be solved by the "method of characteristics"***. The characteristic curves are of course the rays which we have mentioned. They are space-time curves . The characteristic (ordinary) differential equations will be called "ray equations". Thus, by the method of characteristics the phase function can be obtained by integrating an ordinary differential equation along the rays. We also find that the amplitude functions satisfy ordinary differential equations along the rays and that these "transport equations" can be solved. In order to find the rays, and the phase and amplitude functions, it is necessary to specify initial conditions for all of these ordinary differential equations. In some cases (for example, the case of oscillatory initial data) the required initial conditions follow directly from the data of the problem. In others the initial conditions are obtained from the solution of a "canonical problem". A canonical problem is one with the same local features as the given problem. It is, however, sufficiently simple to be solved exactly. The required initial conditions for the given problem are obtained by examination of the asymptotic expansion of the solution of the canonical problem.

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* We hope to complete our research on systems of the form (6) and (26) in the near future.

** For example, problems with constant coefficients and no boundaries.

***See [5], Chapter II.

It is by now clear that the asymptotic method involves several unproved assumptions. It is therefore reasonable to ask whether it can be proved that it does indeed yield the asymptotic expansion of the exact solution of the given problem. No complete proof of this fact has been given. Nevertheless there is abundant evidence of the validity of the method. This evidence has, so far, been obtained in two ways:

1. Problems for (17) in one space dimension can be solved exactly by fourier analysis with respect to t. The resulting ordinary differential equation (with independent variable x) can be solved asymptotically by standard methods for ordinary differential equations (W.K.B. method). Then the asymptotic expansion of the resulting integral representation can be obtained by the "method of stationary phase" and related methods. In every case that has been examined the resulting asymptotic expansion of the exact solution agrees perfectly with the results obtained in this paper by the

"ray method". These investigations yield not only confidence in the validity of the ray method but also an appreciation of the great simplicity of that method when compared with the approach just outlined.*

2. In the case of initial value problems for (6) with oscillatory initial data it is possible to estimate** the remainder of the asymptotic series by energy inequalities. This approach is similar to one outlined by Lax for the initial value problem (3,5), and yields an indirect proof of the asymptotic nature of the expansion in this case.***

The ideas developed in this paper had their origins mainly in the work of J.B. Keller [1,2] and G.B. Whitham [6]. The latter approached the problem of the propagation of dispersive waves from a physical point of view, with considerable success. Our results in sections (3.4) and (4.1) are easily seen to be equivalent to Whitham's results. In attempting to develop Whitham's approach into a systematic expansion procedure, we first introduced the large parameter λ as in (6), (11), (17) and (26,27). This enabled us to proceed in a manner very similar to Keller's method for the reduced wave equation. The essential idea of the "cannonical problem" also had its origins in Keller's work.

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* The author is indebted to Mr. N. Bleistein who has carried out these comparative studies and is currently engaged in other critical examinations of the ray method.

** So far we have been able to obtain estimates in the L_2 norm(energy norm). It is not clear whether pointwise estimates can be obtained by these methods.

*** It is worth noting that such indirect proofs are much more difficult, if not impossible in the case of elliptic equations such as (1) and (4).

Some features of the expansion we shall employ are similar to those of expansions that have been used for the time-dependent Schroedinger equation. In fact the mechanical terms which we shall use such as "hamiltonian", "lagrangian", etc. occur quite naturally when the expansion is applied to the initial value problem for the Schroedinger equation. In this case the appropriate large dimensionless variable is $\lambda = \frac{ap}{h}$ where $\underline{a}$ is a characteristic length $\underline{p}$ is a characteristic momentum and h is Planck's constant. We have found it convenient as well as suggestive to continue to use the mechanical terms for all our equations.

Before proceeding to the asymptotic solution of (17) we add one final remark: If we introduce the transformation

$$x' = \lambda x, \; t' = \lambda t, \; u'(t', x') = u(t,x) \tag{29}$$

(17) becomes

$$u'_{t't'} - [c'(x')]^2 \Delta_{x'} u' + [b'(x')]^2 u' = 0. \tag{30}$$

Here

$$c'(x') = c(x) = c(x'/\lambda), \; b'(x') = b(x) = b(x'/\lambda). \tag{31}$$

Thus, in finding asymptotic solutions for $\lambda \to \infty$ of problems for (17), the transformation (29) shows that we will also be obtaining asymptotic solutions for large t', x' of corresponding problems for equation (30) in which no parameter appears. It should be pointed out, however, that (31) implies that

$$\frac{\partial c'}{\partial x'_\nu} = O(\tfrac{1}{\lambda}), \quad \frac{\partial b'}{\partial x'_\nu} = O(\tfrac{1}{\lambda}). \qquad (32)$$

Therefore, for the case of variable coefficient functions, the asymptotic solutions, as applied to (30) are valid only for slowly-varying coefficient functions. If boundaries are present in the problem for (17) then (29) shows that the corresponding boundaries for (30) must have large dimensions.

In all of the succeeding sections, vectors are denoted by capital letters. We also employ the summation convention with respect to repeated indices.

D2. Asymptotic Series Solutions of a Hyperbolic Equation

Let us assume that a solution $u(t,X) = u(t,x_1,\dots x_\mu)$ of (1.15) has an asymptotic expansion of the form

$$u \sim e^{i\lambda s(t,X)} \sum_m (i\lambda)^{-m} z^{(m)}(t,X); \; z^{(m)} = 0 \text{ for } m = -1,-2,\dots \,. \qquad (1)$$

By inserting (1) in (1.15) and equating to zero coefficients of $(i\lambda)^{-m}$ we obtain the dispersion equation*

$$c^2(X) s_{x_\nu} s_{x_\nu} - s_t s_t + b^2(X) = 0, \qquad (2)$$

and the transport equations

$$2(s_{x_\nu} z^{(m)}_{x_\nu} - c^{-2} s_t z^{(m)}_t) + z^{(m)} \Box s + \Box z^{(m-1)} = 0; \; m=0,1,2,\dots . \qquad (3)$$

Here $\Box s = s_{x_\nu x_\nu} - c^{-2} s_{tt}$.

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*Here the dispersion equation is a partial differential dquation for s whereas in section 1 it was an algebraic equation. No confusion will arise from the double meaning of the term.

We choose a constant reference speed c_o and introduce the following notation, which will be used throughout the paper:

$$x_o = c_o t,\ k_\nu = s_{x_\nu},\ k_o = s_{x_o} = -\frac{\omega}{c_o},\ \omega = -s_t,\ K = (k_1,\ldots,k_\mu),$$
$$\hat{K} = (k_o, K),\ \hat{X} = (x_o, X),\ k^2 = k_\nu k_\nu,\ \hat{k}^2 = k_o^2 + k^2. \tag{4}$$

Then the dispersion equation takes the form

$$c^2k^2 - \omega^2 + b^2 = 0, \tag{5}$$

or, if we solve for ω,

$$\omega = h(X,K) = \pm h_o(X,K);\ h_o(X,K) = \sqrt{c^2k^2+b^2}. \tag{6}$$

For each of the two values of the *hamiltonian*, h, the dispersion equation can be written in the form

$$h(X,K)-\omega = h(X,K)+c_o k_o = 0. \tag{7}$$

We now define the quantities

$$g_\nu = \frac{\partial h}{\partial k_\nu} = \frac{c^2 k_\nu}{\omega},\ g_o = c_o,\ G = (g_1,\ldots,g_\mu),\ \hat{G} = (g_o, G),$$
$$\hat{g}^2 = g_o^2 + g^2,\ g^2 = g_\nu g_\nu = c^2(1-b^2/\omega^2). \tag{8}$$

G is the *group velocity vector*, and g is the *group speed*. By analogy with plane wave solutions of (1.17) we see that it is appropriate to refer to ω and K

as (angular) *frequency* and *propagation vector*, or if we wish to emphasize the mechanical analogy we may refer to K as the *momentum vector*.

In the following it will be noted that we shall make no essential use of the explicit form (6) of the hamiltonian function. For this reason it is not surprising that our methods can be applied to a large class of partial differential equations, as we have asserted in the introduction.

2.1 *the ray equations* The first order partial differential equation (Hamilton-Jacobi equation) (7) can now be solved by introducing the *characteristic equations* (Hamilton's equations of motion)

$$c_o\dot{t} = \dot{x}_o = c_o = g_o, \quad \dot{x}_\nu = \frac{\partial h}{\partial k_\nu} = g_\nu, \quad -\dot{\omega} = c_o\dot{k}_o = 0, \quad \dot{k}_\nu = -\frac{\partial h}{\partial x_\nu}; \quad \nu = 1,\cdots,\mu. \tag{9}$$

Since $\dot{t} = 1$, we may choose the independent variable in this system of ordinary differential equations to be t. Each solution of (9) defines a curve X(t) in X-space or equivalently a space-time curve $[t, X(t)]$ which we shall call a *ray*. From (4) and (9) we now have

$$\dot{s} = \frac{ds}{dt} = s_t + s_{x_\nu}\dot{x}_\nu = \ell \tag{10}$$

where ℓ is the *lagrangian*,

$$\ell = k_\nu\dot{x}_\nu - h = k_\nu g_\nu - \omega = k_\nu\frac{\partial h}{\partial k_\nu} - h. \tag{11}$$

2.2 solution of the dispersion equation

An initial value problem for $s(t,X)$ can now be solved by means of (9) and (11). Let $\Sigma = (\sigma_1, \ldots, \sigma_\mu)$, and let

$$t = t_o(\Sigma),\ x_\nu = x_{\nu o}(\Sigma);\ \nu = 1,\ \ldots,\ \mu \tag{12}$$

be the parametric representation of a hypersurface on which s has the prescribed values

$$s\left[t_o(\Sigma),\ x_{\nu o}(\Sigma)\right] = s_o(\Sigma). \tag{13}$$

Then differentiation of (13) with respect to σ_j yields

$$k_\nu \frac{\partial x_{\nu o}}{\partial \sigma_j} - \omega \frac{\partial t^o}{\partial \sigma_j} = \frac{\partial s_o}{\partial \sigma_j}\ ;\ j = 1,\ \ldots,\ \mu. \tag{14}$$

(14) and (6) provide $\mu+1$ algebraic equations for the determination of $\omega = \omega_o(\Sigma)$ and $k_\nu = k_{\nu o}(\Sigma)$, $\nu = 1, \ldots, \mu$, at each point Σ of the initial hypersurface (12). Then $t_o(\Sigma)$, $x_{\nu o}(\Sigma)$, $\omega_o(\Sigma)$, $k_{\nu o}(\Sigma)$, provide initial values for the solution of (9). If this solution is denoted by $x_\nu = x_\nu(t;\ \Sigma)$, $k_\nu = k_\nu(t;\ \Sigma)$, $\omega = \omega(t;\Sigma) = \omega(\Sigma)$; then $x_\nu\left[t_o(\Sigma);\Sigma\right] = x_{\nu o}(\Sigma)$, $k_\nu\left[t_o(\Sigma);\Sigma\right] = k_{\nu o}(\Sigma)$, and $\omega(\Sigma) = \omega_o(\Sigma)$. In terms of this solution (10) may be integrated to yield

$$s\left[t, x_\nu(t;\Sigma)\right] = s_o(\Sigma) + \int_{t_o(\Sigma)}^{t} \ell\left[x_\nu(t';\ \Sigma),\ k_\nu(t';\Sigma)\right] dt'. \tag{15}$$

Corresponding to the two values of (6), (15) yields the two solutions of the initial value problem for s.

The initial value problem for s is considerably simplified if the initial hypersurface is the hyperplane $t = t_o$. In this case we may determine $k_{\nu o}(\Sigma)$ and $\omega_o(\Sigma)$ at once. We set

$$t_o(\Sigma) = t_o,\ x_{\nu o}(\Sigma) = \sigma_\nu;\ \nu = 1, \ldots, \mu; \tag{16}$$

in (12). Then (14) and (6) yield

$$k_{jo}(\Sigma) = \frac{\partial s_o}{\partial \sigma_j};\quad \omega_o(\Sigma) = h(\sigma_\nu, \frac{\partial s_o}{\partial \sigma_\nu}) = \pm\sqrt{c^2(\Sigma)\frac{\partial s_o}{\partial \sigma_\nu}\frac{\partial s_o}{\partial \sigma_\nu} + b^2(\Sigma)}. \tag{17}$$

It is interesting to note that (11) implies that s is constant on rays if and only if $k_\nu \frac{\partial h}{\partial k_\nu} = h$. But this is equivalent to the condition that h be homogeneous of degree one in K. In this case the original partial differential equation contains no lower order terms and is non-dispersive. For non-dispersive equations it is easily seen that the dispersion equation (as a differential equation for s) is the "characteristic equation", the level surfaces of its solution s(t,X) are characteristic hyper-surfaces of the original equation, and the rays lie on the characteristic surfaces. None of these statements is true for dispersive equations such as (1.17).

2.3 solution of the transport equations

We turn our attention now to the transport equations (3) which may be written in the form

$$\dot z^{(m)} + (\frac{c^2}{2\omega}\Box s)z^{(m)} = z_t^{(m)} + g_\nu z_{x_\nu}^{(m)} + (\frac{c^2}{2\omega}\Box s)z^{(m)} = -\frac{c^2}{2\omega}\Box z^{(m-1)};\ m=0,1,2,\ldots . \tag{18}$$

If we set $z^{(0)} = z$, the zero order equation is

$$z + (\frac{c^2}{2\omega}\Box s)z = z_t + g_\nu z_{x_\nu} + (\frac{c^2}{2\omega}\Box s)z = 0. \tag{19}$$

These are ordinary differential equations along each ray. To solve the homogenous equation (19) we first note that

$$\Box\, s = s_{x_\nu x_\nu} - c^{-2} s_{tt} = c^{-2}\omega_t + \frac{\partial k_\nu}{\partial x_\nu} = \frac{\partial}{\partial x_0}\left(\frac{\omega}{c^2} c_0\right) + \frac{\partial}{\partial x_\nu}\left(\frac{\omega}{c^2} g_\nu\right) = \hat{\nabla}\cdot\left(\frac{\omega}{c^2}\hat{G}\right) \tag{20}$$

where

$$\hat{\nabla} = \left(\frac{\partial}{\partial x_0}, \nabla\right), \quad \hat{G} = (g_0, G), \; g_0 = c_0 \; . \tag{21}$$

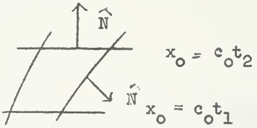
Then (19) can be written in the form

$$\frac{2\omega}{c^2}\, \hat{G}\cdot\hat{\nabla} z + z\hat{\nabla}\cdot\left(\frac{\omega}{c^2}\hat{G}\right) = 0, \tag{22}$$

or

$$\hat{\nabla}\cdot\left(\frac{z^2\omega}{c^2}\hat{G}\right) = 0 \tag{23}$$

For an arbitrary domain D in Σ-space we now apply Gauss' theorem to the "ray-tube" in $\hat{X}$-space determined by: Σ in D, $t_1 \leq t \leq t_2$:



Since $\hat{G} = (c_o, G)$ is everywhere parallel to the rays $\hat{X} = (x_o, X) = [c_o t, X(t,\Sigma)]$, (23) yields

$$0 = \int \hat{\nabla}\cdot\left(\frac{z^2\omega}{c^2}\hat{G}\right) dx_o \ldots dx_\mu = \int \frac{z^2\omega}{c^2}\hat{G}\cdot\hat{N} dS = \left\{ \int\limits_{t=t_2} - \int\limits_{t=t_1} \right\} \frac{\omega z^2 c_o}{c^2}\, dx_1 \ldots dx_\mu$$

$$= \left\{ \int\limits_{\substack{t=t_2 \\ (\Sigma \text{ in } D)}} - \int\limits_{\substack{t=t_1 \\ (\Sigma \text{ in } D)}} \right\} \frac{\omega z^2}{c^2}\, c_o |j|\, d\sigma_1 \ldots d\sigma_\mu . \tag{24}$$

Here j is the jacobian,

$$j(t,\Sigma) = \frac{\partial(x_1, \ldots, x_\mu)}{\partial(\sigma_1, \ldots, \sigma_\mu)} = \det\left[\frac{\partial x_\nu(t;\Sigma)}{\partial\sigma_1}\right]. \tag{25}$$

Since D was arbitrary and ω is constant on a ray (24) implies that $\frac{z^2 j}{c^2}$ is constant on a ray (provided j does not vanish). Thus the solution of (19) is given by

$$z(t) = z(t_o) \frac{c(t)}{c(t_o)} \left[\frac{j(t_o)}{j(t)}\right]^{1/2}. \tag{26}$$

Here $z(t) = z[t, X(t,\Sigma)]$, $c(t) = c[X(t,\Sigma)]$, and $j(t) = j(t,\Sigma)$. The solution of the inhomogeneous equation (18) can now be obtained by the method of "variation of parameters". The result is

$$z^{(m)}(t) = z^{(m)}(t_o) \frac{c(t)}{c(t_o)} \left[\frac{j(t_o)}{j(t)}\right]^{1/2} - \frac{c(t)}{2\omega} \int_{t_o}^{t} \left[\frac{j(t')}{j(t)}\right]^{1/2} c(t') \Box z^{(m-1)}(t') dt'; \tag{27}$$

$$m = 0,1,2,\ldots.$$

2.4 Constant coefficients

The foregoing results simplify considerably in case the coefficient functions are constant: $b = b_o$, $c = c_o$. In this case the ray equations (9) imply that k_ν, ω, g_ν, and ℓ are constant on each ray; i.e.

$$k_\nu = k_{\nu o}(\Sigma),\ \omega = \omega_o(\Sigma),\ g_\nu = g_{\nu o}(\Sigma) = h_{k\nu}\left[k_{jo}(\Sigma)\right],$$

$$\ell = \ell_o(\Sigma) = k_{\nu o}\, g_{\nu o} - \omega_o. \tag{28}$$

Furthermore the rays are the straight lines

$$x_\nu = x_{\nu o}(\Sigma) + g_{\nu o}(\Sigma)\,[t-t_o(\Sigma)] = x_{\nu o} - t_o g_{\nu o} + g_{\nu o} t, \tag{29}$$

and

$$s = s_o(\Sigma) + \ell_o(\Sigma)[t-t_o(\Sigma)] = s_o - t_o\,\ell_o + \ell_o t. \tag{30}$$

(25) and (29) now imply that (in general)

$$j(t,\Sigma) = \det\left[(x_{\nu o} - t_o g_{\nu o})_{\sigma_i} + (g_{\nu o})_{\sigma_i} t\right] \tag{31}$$

is a polynomial of degree μ in t, with coefficients which are functions of Σ. Hence if $\tau_1(\Sigma),\ldots,\tau_\mu(\Sigma)$ are the roots of this polynomial we may write

$$j(t,\Sigma) = a(\Sigma)\prod_{\nu=1}^{\mu}\,[t-\tau_\nu(\Sigma)], \tag{32}$$

and (27) takes the special form

$$z^{(m)}(t) = z^{(m)}(t_o)\left[\frac{j(t_o)}{j(t)}\right]^{1/2} - \frac{c_o^{\,2}}{2\omega_o}\int_{t_o}^{t}\left[\frac{j(t')}{j(t)}\right]^{1/2} \Box z^{(m-1)}(t')dt';\ m = 0,1,2,\ldots, \tag{33}$$

where

$$\frac{j(t_o)}{j(t)} = \frac{(t_o-\tau_1)\ldots,(t_o-\tau_\mu)}{(t-\tau_1)\ldots(t-\tau_\mu)}\ . \tag{34}$$

2.5 plane waves

The simplest examples of our formal expansion are the plane waves. We assume that the coefficients are constant and take the initial surface to be the hyperplane $t = 0$. For arbitrary constant $K = (k_1,\ldots,k_\mu)$ we set $s^o(\Sigma) = k_\nu \sigma_\nu$. Then (17) yields

$$k_{\nu o}(\Sigma) = k_\nu \, , \; \omega_o(\sigma) = h(k_\nu) = \pm\sqrt{c_o^2 \, k^2 + b_o^2} \; . \tag{35}$$

We define $g_{\nu o}$ and ℓ_o by (28). Then (29), yields the family of parallel straight rays

$$x_\nu = \sigma_\nu + g_{\nu o} t \tag{36}$$

and (30) yields

$$s = k_\nu \sigma_\nu + (k_{\nu o} g_{\nu o} - \omega_o) t = k_\nu x_\nu - \omega_o t. \tag{37}$$

From (36) we see that $j \equiv 1$. We now set $z^{(m)}$ = const. on the plane $t = 0$ for $m = 0,1,2,\ldots$. Then (33) implies that $z^{(m)}(t,X)$ is constant. If we insert these results in (1) we obtain

$$u(t,X) = z(\lambda) e^{i\lambda(k_\nu x_\nu - \omega_o t)}; \; z(\lambda) \sim \sum (i\lambda)^{-m} z^{(m)}. \tag{38}$$

In fact (38) is an exact solution of (1.15) with arbitrary $z(\lambda)$ provided ω_o satisfies (35).

2.6 waves

Any asymptotic series solution of (1.17) of the form (1) will be called a wave. Plane waves are a special case. We have seen that if the value of s is prescribed on a suitable μ-dimensional manifold M, two solutions s of the dispersion equation are determined (at least in a neighborhood of M). If the values of the functions $z^{(m)}$ for $m = 0,1,2,\ldots$ are also prescribed on M exactly two waves are completely determined. In succeeding sections we will attempt to show how a sum of one or more waves can be constructed to yield asymptotic solutions of suitable problems for (1.17).

2.7 lower dimensional initial manifolds

For some problems it will be necessary to construct waves which are determined by the values of s and $z^{(m)}$ on a manifold M of dimension $\delta < \mu$. In the extreme case, $\delta = 0$, M is a single point $(t_o, X_o) = (t_o, X_{1o}, \ldots, X_{\mu o})$, and s has the prescribed value

$$s(t_o, X_o) = s_o. \tag{39}$$

We shall then require waves for which all of the assosciated rays pass through M. These rays are obtained by solving the ray equations with initial conditions

$$x_\nu(t_o) = x_{\nu o},\ k_\nu(t_o) = p_\nu\ ,\ \omega(t_o) = h(X_o, P) = \pm\ h_o(X_o, P). \tag{40}$$

Here $P = (p_1, \ldots, p_\mu)$ is arbitrary, and we obtain the μ-parameter family of rays, $x_\nu = x_\nu(t;P)$ for which $x_\nu(t_o;\ P) = x_{\nu o}$. (In fact we obtain two such families, one corresponding to each of the two values of h). When the rays are determined, s(t,X) is given, on each ray by

$$s(t) = s\left[t, X(t;\ P)\right] = s_o + \int_{t_o}^{t} \ell\left[X(t';\ P),\ P\right]dt'. \tag{41}$$

Here $\ell(X,K) = k_\nu\, h_{k_\nu} - h$.

In solving the transport equations to determine the functions $z^{(m)}$ along the rays emanating from M, our earlier argument is valid if we replace $\Sigma = (\sigma_1, \ldots, \sigma_\mu)$ everywhere by $P = (p_1, \ldots, p_\mu)$ and replace t_o in (26) and (27) by t_1, where $t_o < t_1$. Of course now

$$j = j(t,P) = \frac{\partial(x_1,\ldots,x_n)}{\partial(p_1,\ldots,p_n)} = \det\left[\frac{\partial x_\nu(t;P)}{\partial p_i}\right]. \tag{42}$$

However it is clear that j vanishes at the point M, i.e. at $t = t_o$, hence the functions $z^{(m)}$ become singular there, which is what we expect at a caustic. If we set

$$\tilde{z} = \lim_{t_1\to t_o} \frac{z(t_1)|j(t_1)|^{1/2}}{c(t_1)} \tag{43}$$

then, we may let $t_1 \to t_o$ in (27) (for m = 0). Thus we obtain

$$z(t) = \tilde{z}\, c(t)|j(t)|^{-1/2}. \tag{44}$$

Here $z(t) = z[t, X(t;P)]$, $c(t) = c[X(t;P)]$, $j(t) = j(t,P)$, $\omega = h(X_o,P)$, and $\tilde{z} = \tilde{z}\ (P)$. At the beginning of the preceeding paragraph we stated that the waves associated with the point manifold M are determined by the values of s and $z^{(m)}$ on M. We see now that this statement must be modified, for the $z^{(m)}$ are infinite on M. In fact, the leading term of each of the two waves emanating from M is uniquely determined by the values of the number s_o and the function $\tilde{z}(P)$.

For the case of constant coefficients, k_ν is constant on each ray, i.e. $k_\nu = k_\nu(t_o) = p_\nu$, and the rays are given by

$$x_\nu = x_{\nu o} + (t-t_o)g_\nu;\ g_\nu = h_{k_\nu}(P) = h_{k_\nu}(K) \tag{45}$$

Furthermore (41) becomes

$$s(t) = s_o + (t-t_o)\ell \;;\; \ell = k_\nu h_{k_\nu} - h = \ell(K), \tag{46}$$

and

$$j(t,K) = \det\left|(t-t_o)h_{k_\nu k_i}\right| = (t-t_o)^\mu \det(h_{k_\nu k_i}). \tag{47}$$

Finally, we may replace (46) by

$$z(t) = c\,\tilde{z}\left|(t-t_o)^\mu \det(h_{k_\nu k_i})\right|^{-1/2}. \tag{48}$$

It is of interest to note that the discussion of this subsection is not valid for a non-dispersive equation. For such equations the rays emanating from a point all lie on the surface of the characteristic conoid determined by that point, hence the rays do not fill a region of t,X-space and the concept of a "tube" becomes meaningless. Furthermore, the factor $\det(h_{k_\nu k_i})$ in (48) vanishes in the non-dispersive case.*

3 . Initial-boundary Value Problems for the Homogeneous Equation

In this section we will show how sums of one or more waves can be constructed to yield asymptotic solutions of initial and initial-boundary value problems for (1.17). We restrict our attention to the simple case of one space dimension ($\mu=1$) and shall obtain just the leading term of the asymptotic expansion. Our examples are intended to be only illustrations of methods which are applicable to higher dimensional problems as well as to other differential equations. Therefore we shall avoid using special methods which are suitable only for the particular equation under consideration.

We consider the following initial value problem:

$$c^2(x)u_{xx} - u_{tt} - \lambda^2 b^2(x)u = 0, \tag{1}$$

$$u(0,x) = u_o(x;\lambda), \quad u_t(0,X) = u_1(x;\lambda). \tag{2}$$

---

* This can be seen by differentiating the Euler equation $k_\nu \frac{\partial h}{\partial k_\nu} = h$ with respect to k_i.

In some cases we shall also prescribe a homogeneous boundary condition. An interesting feature of the asymptotic method is that entirely different types of solutions are obtained, depending upon how the parameter λ appears in the initial data u_0, u_1. We consider first the initial value problem with the simplest type of initial data.

3.1 Oscillatory initial data

Let us choose initial data which already have the form of a wave. If $u(t,x) \sim z(t,x)e^{i\lambda s(t,x)}$ then $u_t(t,x) \sim i\lambda s_t z e^{i\lambda s} = -i\lambda h(s_x) z\, e^{i\lambda s}$ where $h(k) = \pm\, h_0(k) = \pm\sqrt{c^2k^2+b^2}$. Let us therefore consider the initial value problem (1,2) with

$$u_0(x;\lambda) \sim z_0(x)e^{i\lambda s_0(x)},\quad u_1(x;\lambda) \sim -i\lambda h[s_0'(x)]z_0(x)e^{i\lambda s_0(x)}. \tag{3}$$

(Actually we are simultaneously considering two initial value problems, since $h(k)$ has two values.) The functions z_0 and s_0 are given.

If we attempt to solve this initial value problem by constructing a wave,

$$u(t,x) \sim z(t,x)e^{i\lambda s(t,x)} \tag{4}$$

we see that (3) yields the values of s and z at $t = 0$:

$$s(0, \sigma) = s_0(\sigma) \quad\text{and}\quad z(0,\sigma) = z_0(\sigma). \tag{5}$$

Then $z(t,x)$ and $s(t,x)$ can be determined by the method of section 2. Let us carry out the details for the case of constant coefficients. From (2.17)

$$k_0(\sigma) = s_0'(\sigma);\ \omega_0(\sigma) = h\left[s_0'(\sigma)\right]. \tag{6}$$

We then solve the ray equations with $k_o(\sigma)$, $\omega_o(\sigma)$ given by (6), $t_o(\sigma) = 0$, and $x_o(\sigma) = \sigma$. From (2.30) and (2.31) we see that the ray emanating from the point $(t,x) = (0,\sigma)$ is given by

$$x = \sigma + gt; \quad g = h'\left[s_o'(\sigma)\right]; \tag{7}$$

while on this ray

$$k = s_o'(\sigma), \ \omega = h\left[s_o'(\sigma)\right], \quad s = s_o(\sigma) + \ell t; \quad \ell = kg - \omega. \tag{8}$$

Here $h' = dh/dk$.

To determine $z(t,x)$ we first compute the jacobian,

$$j(t,\sigma) = \frac{dx}{d\sigma} = 1 + t\frac{dg}{d\sigma} = 1 + th''(s_o')s_o'' \ . \tag{9}$$

Then $j(0,\sigma) = 1$, and (2.35) yields the value of z on the ray:

$$z(t) = z_o(\sigma)\left[1 + th''(s_o')s_o''\right]^{-1/2}. \tag{10}$$

Thus we see that the asymptotic solution of the initial value problem is given parametrically (with parameter σ) by

$$u \sim z_o(\sigma)\left\{1 + th''\left[s_o'(\sigma)\right]s_o''(\sigma)\right\}^{-1/2}\exp\left\{i\lambda\left(s_o(\sigma) + \left[s_o'\ h'(s_o') - h(s_o')\right]t\right)\right\},$$

$$x = \sigma + h'\left[s_o'(\sigma)\right]t \ ; \ h(k) = \pm\, h_o(k) = \pm\sqrt{c^2k^2 + b^2}. \tag{11}$$

Now

$$h_o'(k) = c^2(c^2k^2 + b^2)^{-1/2}k \ ; \quad h_o''(k) = c^2b^2(c^2k^2 + b^2)^{-3/2} > 0. \tag{12}$$

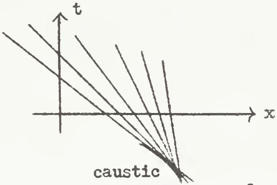
Let us assume for the sake of definiteness that

$$s_o''(\sigma) \leq 0. \tag{13}$$

Let us also denote the solutions corresponding to the two values $h = \pm h_o$ by $u_\pm$. Then for u_- we see from (12) and (13) that the jacobian (10) vanishes only for $t < 0$, and the solution formula (11) for u_- does not become singular for $t > 0$. However it is easy to see that

$$u_-(t,x) = u_+(-t,x) \tag{14}$$

hence the representation (11) for u_+ becomes singular for certain points in the half-space $t > 0$. It is easy to show that these points lie on the caustic curve or envelope of the ray system. The singularity is a defect of the asymptotic solution (11) and must be repaired by considering a different kind of asymptotic expansion in the neighborhood of the caustic. The rays of u_- are illustrated in the following figure. The corresponding picture for u_+ is obtained by reflection in the x axis.



It is interesting to note that if $m = dt/dx = g^{-1}$ is the slope of a ray, then from (12) $|m| > c^{-1}$, i.e. c is an upper bound on the group speed.

Let us now set

$$z_o(x) = -\frac{y_o(x)}{h_o[s_o'(x)]} \tag{15}$$

and

$$u(t,x) = \frac{1}{2}[u_+(t,x) - u_-(t,x)]. \tag{16}$$

Then u is an asymptotic solution of (1) and we see from (3) that it satisfies the initial conditions

$$u(0,x) \sim 0; \quad u_t(0,x) \sim i\lambda y_o(x)e^{i\lambda s_o(x)}. \tag{17}$$

Thus the asymptotic solution of the initial value problem with initial conditions (17) consists of a sum of two waves. Finally, let us set

$$u(t,x) = u_2(t,x) + \frac{1}{i\lambda}\frac{\partial}{\partial t} u_1(t,x) \tag{18}$$

where u_1 and u_2 are solutions with initial data

$$u_j(0,x) \sim 0, \qquad \frac{\partial}{\partial t} u_j(0,x) \sim i\lambda y_j(x)e^{i\lambda s_j(X)}; \ j = 1,2. \tag{19}$$

Then u is the solution of (1) which satisfies the initial conditions

$$u(0,x) \sim y_1(x)e^{i\lambda s_1(x)}, \ u_t(0,x) \sim i\lambda y_2(x)e^{i\lambda s_2(x)}. \tag{20}$$

(In deriving (20) we have used the fact that, at $t = 0$, $u_{1tt} = c^2u_{1xx} - \lambda^2b^2u_1 \sim 0$.) Thus we see that the asymptotic solution of the initial value problem with oscillatory initial data (20) can, in general, be obtained as a sum of waves.

It is interesting to note the special features of'our result when $b = 0$ In this case the parameter λ is absent from the partial differential equation (1) (in fact that equation becomes merely the wave equation) and appears only in the initial data. From (12) we see that the group speed $g = h'(k)$ becomes $g = \pm c$, and the rays lie on the characteristic surfaces $x \pm ct =$ constant. In this case the expsnion reduces to Lax's expansion [4].

3.2 the phase-shift at a caustic

In appendix I the exact solution of the initial value problem (1,2) (with constant coefficients) is given. Let us set $h = + h_o$ in (3) and examine the solution u_+ by the method of appendix I. From (I.10) we see that

$$u_+(t,x) \sim \frac{\lambda}{4\pi} \sum_{\pm} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\sigma \left\{ 1 \pm \frac{h_o[s'_o(\sigma)]}{h_o(k)} \right\} z_o(\sigma) e^{i\lambda\phi_{\pm}(k,\sigma)} \tag{21}$$

where

$$\phi_{\pm}(k,\sigma) = k(x-\sigma) \mp h_o(k)t + s_o(\sigma). \tag{22}$$

We now apply the stationary phase formula (II. 12) of appendix II to obtain the asymptotic expansion of (21) for $\lambda \to \infty$. The stationary point is determined by the conditions

$$\frac{\partial}{\partial k} \phi_{\pm} = x-\sigma \mp h'_o(k)t = 0; \quad \frac{\partial}{\partial \sigma} \phi_{\pm} = s'_o(\sigma)-k = 0. \tag{23}$$

At the stationary point $h_o[s'_o(\sigma)]/h_o(k) = 1$ therefore the term corresponding to the lower sign in (21) vanishes. At the stationary point (for the term with the upper sign) (22) and (23) yield

$$x = \sigma + h'_o[s'_o(\sigma)]t \ , \quad k = s'_o(\sigma), \tag{24}$$

$$\phi = \phi_+ = s_o(\sigma) + \left\{ s'_o(\sigma)\, h'_o[s'_o(\sigma)] - h_o[s'_o(\sigma)] \right\} t, \tag{25}$$

and

$$(\phi_{\nu j}) = \begin{pmatrix} \phi_{kk} & \phi_{k\sigma} \\ \phi_{\sigma k} & \phi_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} -h_0''t & -1 \\ -1 & s_0'' \end{pmatrix}; \; h_0'' = h_0''[s_0'(\sigma)]. \tag{26}$$

Hence $$\det(\phi_{\nu j}) = -(1 + h_0''\, s_0''t). \tag{27}$$

It follows now from (II.12) that

$$u_+ \sim z_0(\sigma) \left\{ \left|1 + th_0''[s_0'(\sigma)]s_0''(\sigma)\right| \right\}^{-1/2} \exp\left\{ i\lambda\left(s_0(\sigma) + \left[s_0'h_0'(s_0') - h_0(s_0')\right]t\right) \right.$$

$$\left. + i\tfrac{\pi}{4}\, \mathrm{sig}(\phi_{\nu j}) \right\},$$

$$x = \sigma + h_0'[s_0'(\sigma)]t \quad , \quad h_0(k) = \sqrt{c^2k^2 + b^2} . \tag{28}$$

In some neighborhood of the initial plane $t = 0$, $|1 + th_0''\, s_0''| = 1 + th_0''s_0''$,

and we see that (28) agrees exactly with (11) (for $h = + h_0$) provided $\mathrm{sig}(\phi_{\nu j}) = 0$.

In order to determine $\mathrm{sig}(\phi_{\nu j})$ we compute the eigenvalues r_1, r_2 of $(\phi_{\nu j})$. They are the roots of the polynomial

$$\det(\phi_{\nu j} - r\delta_{\nu j}) = (-h_0''t - r)(s_0'' - r) - 1 = r^2 + (h_0''t - s_0'')r - s_0''h_0''t - 1. \tag{29}$$

Thus

$$2r_{\substack{1\\2}} = (s_0'' - h_0''t) \pm \left[(s_0'' + h_0''t)^2 + 4\right]^{1/2}. \tag{30}$$

In a neighborhood of $t = 0$ it is now clear that $r_1 > 0$, $r_2 < 0$; hence $\mathrm{sig}\,(\phi_{\nu j}) = \mathrm{sgn}\, r_1 + \mathrm{sgn}\, r_2 = 0$. As a function of t, $\mathrm{sig}(\phi_{\nu j})$ changes discontinuously when (27) vanishes, i.e. when

$$t = t_o(\sigma) = -(h_o'' \, s_o'')^{-1} \tag{31}$$

We recall that $h_o'' > 0$ and $s_o'' < 0$. Therefore $t_o(\sigma) > 0$. From (30) we see that at $t = t_o$,

$$2r_{\substack{1\\2}} = \left(s_o'' + \frac{1}{s_o''} \right) \pm \left[\left(s_o'' - \frac{1}{s_o''} \right)^2 + 4 \right]^{1/2} = \left(s_o'' + \frac{1}{s_o''} \right) \pm \left| s_o'' + \frac{1}{s_o''} \right| = \left(s_o'' + \frac{1}{s_o''} \right) \mp \left(s_o'' + \frac{1}{s_o''} \right). \tag{32}$$

Therefore at $t = t_o$ r_1 vanishes while r_2 remains negative. For $t > t_o$ both roots are negative. Thus

$$\operatorname{sig}(\phi_{\nu j}) = \left\{ \begin{array}{lll} 0 & \text{for} & t < t_o \\ -2 & \text{for} & t > t_o \end{array} \right\}. \tag{33}$$

The result (11) for u_+ which was obtained without the use of the solution formula (I.10) is valid in the region $t < t_o(\sigma)$. We now see that that result is verified by (28) and (33). However (28) and (33) are also valid for $t > t_o(\sigma)$. This suggests a rule for extending our general method to the region $t > t(\Sigma)$, beyond a caustic $t = t(\Sigma)$: The formula (2.28) for the function z becomes indeterminate for $t > t(\Sigma)$ because $j(t_1)/j(t)$ becomes negative. The indeterminacy is removed by replacing j/j by $|j/j|$ and introducing the phase-shift factor $\exp\left\{-i\pi/2\right\}$. This "phase-shift rule" is identical to an analogous rule for the reduced wave equation [9].

In section 3.1 we noted that the asymptotic formula fails (i.e. becomes infinite) at the caustic $t = t(\sigma)$. The correct expansion at $t = t(\sigma)$ could be obtained by expanding the integral (21) by means of a different stationary phase formula which is valid when $\det(\phi_{\nu j})$ vanishes at the stationary point.

3.3 A simple initial-boundary value problem

The initial value problem for u_-, required the solution of (1) with the initial conditions

$$u(0,x) \sim z_o(x)e^{i\lambda s_o(x)}, \; \frac{\partial u}{\partial t}(0,x) \sim i\lambda h_o[s_o'(x)]z_o(x)e^{i\lambda s_o(x)} \tag{34}$$

The asymptotic solution was given by (11) with $h = -h_o$. Let us now modify the problem by adding the boundary condition at $x = 0$,

$$u(t,0) \equiv 0. \tag{35}$$

The solution in the quarter-space $t \geq 0$, $x \geq 0$ is required. For consistency we must assume that $z_o(0) = 0$.

The asymptotic solution of this initial-boundary value problem is obtained by adding to u_- a reflected wave $u_r(t,x)$ which, together with u_- satisfies (35). The reflected wave must vanish at $t = 0$ so that (34) is unchanged. Thus we set

$$u = u_- + u_r; \; u_-(t,x) \sim z(t,x)e^{i\lambda s(t,x)}; \; u_r(t,x) \sim z_r(t,x)e^{i\lambda s_r(t,x)}. \tag{36}$$

Here z and s are given along the "incident" rays (7) by (10) and (8).

By inserting (36) in (35) we obtain

$$s_r(t,0) = s(t,0), \tag{37}$$

$$z_r(t,0) = -z(t,0), \tag{38}$$

and differentiation of (37) with respect to t yields

$$\omega_r(t,0) = \omega(t,0). \tag{39}$$

Since both incident and reflected waves satisfy the dispersion equation $\omega^2 = c^2k^2+b^2$ we conclude that $k_r(t,0) = \pm k(t,0)$, hence $g_r(t,0) = \pm g(t,0)$. In order that the reflected rays should not intersect the initial line,* $t = 0$, $x \geq 0$, we must take

$$k_r(t,0) = -k(t,0), \; g_r(t,0) = -g(t,0). \tag{40}$$

It is convenient to express the boundary parametrically in terms of the same parameter, σ, used in describing the incident wave u . From (7) we see that at $x = 0$, $\sigma + g(\sigma)t = 0$. Hence the boundary is given parametrically by

$$x = x_o(\sigma) = 0 \quad , \quad t = t_o(\sigma) = -\sigma/g(\sigma). \tag{41}$$

Then, on the reflected ray emanating from the boundary at $[t_o(\sigma),0]$, (39), (40), and (6-8) yield

$$\begin{aligned}\omega_r = \omega = -h_o[s_o'(\sigma)],\; k_r = -k = -s_o'(\sigma),\; g_r = -g = h_o'[s_o'(\sigma)],\; \ell_r = k_r g_r - \omega_r \\ = kg - \omega = \ell\end{aligned} \tag{42}$$

From (2.31) the parametric equation of the reflected ray is

$$x = g_r(t-t_o) = -g(t-t_o) = -\sigma - gt = -\sigma + th_o'[s_o'(\sigma)], \tag{43}$$

while (2.32), (37), (8), and (42) yield

$$s_r = s[t_o(\sigma)] + \ell_r(t-t_o) = s_o(\sigma) + \ell t_o + \ell(t-t_o) = s_o(\sigma) + \ell t. \tag{44}$$

---

*This condition is clearly required if $u = u_- + u_r$ is to satisfy the initial conditions, for then we must require that $u_- = 0$ at $t = 0$, $x \geq 0$.

From (43) we may now compute the jacobian,

$$j = \frac{dx}{d\sigma} = - \left\{ 1 - th_0'' \left[s_0'(\sigma) \right] s_0''(\sigma) \right\} . \tag{45}$$

Then (38), (2.28), and (10) yield

$$\begin{aligned} z^r(t) = z\left[t_0(\sigma)\right] \left[\frac{j(t_0)}{j(t)}\right]^{1/2} &= -z_0(\sigma) \left[1-t_0 h_0'' \, s_0''\right]^{-1/2} \left[\frac{1-t_0 h_0'' \, s_0''}{1-th_0'' s_0''}\right]^{1/2} \\ &= -z_0(\sigma)\left[1-t \, h_0'' \, s_0''\right]^{-1/2}. \end{aligned} \tag{46}$$

Summarizing our results, we see that $u = u_- + u_r$ where

$$u_r \sim -z_0(\sigma)\left[1-th_0'' \, s_0''\right]^{-1/2} \exp\left\{ i\lambda(s_0 + \ell \, t) \right\} , \quad x = -\sigma-gt \tag{47}$$

and

$$u_- \sim z_0(\sigma)\left[1-th_0'' \, s_0''\right]^{-1/2} \exp\left\{ i\lambda(s_0 + \ell \, t) \right\} , \quad x = \sigma+gt. \tag{48}$$

This completes the solution of the initial-boundary value problem.

From (47) and (48) we note that

$$u_r(t,x) = -u_-(t,-x). \tag{49}$$

Of course (49) is easily obtained by the "image method"* which yields the solution of the initial-boundary value problem at once in terms of the solution u_- of the initial value problem. Thus this method provides a check on the construction we have used.

The above problem, though essentially trivial suffices to illustrate the general procedure for the asymptotic solution of initial-boundary value problems: We first obtain the asymptotic solution of the initial value problem in terms of one or more waves emanating from a space-like initial surface. When these waves are "incident" on a time-like boundary, additional reflected waves are produced. Each reflected wave is uniquely determined by the incident wave and the boundary condition. Of course we must expect complications to occur when incident rays are tangent to the boundary or when boundaries are not smooth. In such cases there will be "shadows" and we shall have to study "diffracted" waves in the shadow region.*

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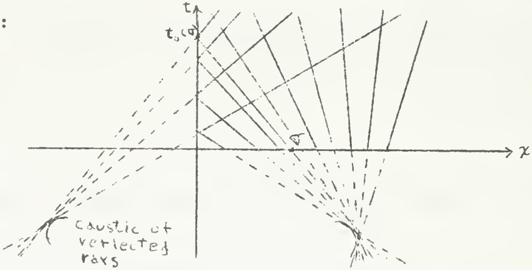
* In this case the image method would consist in extending the initial data to $x < 0$ in such a way that $u(0,x)$ and $u_t(0,x)$ are odd function of x.

The incident and reflected rays of our simple problem are illustrated in the following figure:



3.4 rapidly varying initial data

Our results so far have been restricted to problems with oscillatory initial data. In this, and succeeding sections we shall consider other kinds of initial data. In each case the initial data give rise to one or more waves. Then if the problem involves boundaries, reflected waves will be produced just as in section 3.3. Thus the essential new features are embodied in problems without boundaries and we may restrict our attention to pure initial value probelms.

---

*Such diffracted waves for elliptic problems are exhaustively treated in [1] and [2].

In this section we consider the initial value probelm for equation (1) with initial data of the form

$$u(0,x) = u_0(x; \lambda) = u_0(\lambda x),\ u_t(0,x) = u_1(x;\lambda) = \lambda u_1(\lambda x). \tag{50}$$

Such data are called rapidly varying because of their large derivatives. In terms of the transformation (1.31) this problem is equivalent to the problem for the differential equation without a parameter,

$$(c')^2\, u'_{x'x'} - u'_{t't'} - (b')^2 u' = 0 \tag{51}$$

and with initial data of the form

$$u'(0,x') = u_0(x'),\ u'_{t'}(0,x') = u_1(x'), \tag{52}$$

i.e. also independent of the parameter. Thus we see that rapidly varying initial data are quite natural for certain problems.

We assume that the functions $u_0(x')$, $u_1(x')$ have compact support, i.e. vanish outside a finite region called the support of these functions. Then for $\lambda \to \infty$ the support of $u_0(\lambda x)$ and $\lambda u_1(\lambda x)$ shrinks to the point $x = 0$. This suggests that the waves produced by the rapidly varying initial data (50) emanate from the single point (0,0). In section (2.7) we saw that a point initial manifold, in general, gives rise to two waves. Therefore we set

$$u = u_+ + u_-;\quad u_\pm \sim z_\pm\, e^{i\lambda s_\pm}\ . \tag{53}$$

The functions $z_\pm$ and $s_\pm$ are given by the method of section 2.7 except that the numbers $s_{0\pm}$ and the functions $\tilde{z}_\pm(p)$ are undetermined. In order to determine them we proceed as follows: First we complete the construction for the case of constant coefficients. Then we shall compare the result with the asymptotic expansion of the exact solution of our problem. This will determine $s_{0\pm}$ and $\tilde{z}_\pm(p)$. We then postulate that the newly determined quantities have the same values for the case of variable coefficients*.

From (2.47) the rays are given by

$$x = gt; \; g = h'(k) = \pm h'_0(k). \tag{54}$$

Then from (2.48) and (2.50) we see that $s_\pm$ and $z_\pm$ are given along the rays by

$$s_\pm(t) = s_{0\pm} + \ell t; \quad \ell = kh'(k)-h, \tag{55}$$

and

$$z_\pm(t) = \tilde{z}_\pm(k)c\,|t\, h''_0(k)|^{-1/2} \tag{56}$$

Therefore for $t > 0$, $u(t,x)$ is given parametrically in terms of the parameter k by

$$u \sim \sum_{\pm} c\tilde{z}_\pm(k)\,|th''_0(k)|^{-1/2} \exp\left\{ i\lambda\left[s_{0\pm} \pm (kh'_0 - h_0)t\right]\right\}; \qquad x = \pm h'_0(k)t. \tag{57}$$

The exact solution of the initial value problem (1,50) with constant coefficients is given by (I.3) with $a_\pm$ given by (I.9). We therefore insert (50) in (I.9). This yields

---

*This is the method of comparison with the asymptotic expansion of the solution of a "cannonical problem". The validity of the procedure for the problems treated here has been confirmed, as pointed out in the introduction.

$$a_{\pm}(k) = \frac{\lambda}{4\pi}\int\left[u_0(\lambda x) \pm \frac{i}{h_0(k)}\, u_1(\lambda x)\right]e^{-i\lambda kx}dx = \frac{1}{4\pi}\int\left[u_0(x') \pm \frac{i}{h_0(k)}\, u_1(x')\right]e^{-ikx'}dx'. \tag{58}$$

We note that $a_{\pm}$ is independent of λ. From (I.3),

$$u(t,x) = \sum_{\pm}\int a_{\pm}(k)e^{i\lambda\phi(k)}dk; \quad \phi = kx-ht;\ h = \pm h_0 = \pm\sqrt{c^2k^2+b^2}\ . \tag{59}$$

We now apply the one-dimensional stationary phase formula (II.15) to each of the two terms in (59). The stationary point is determined by

$$\phi'(k) = x-gt = 0, \quad g = h'(k) = \pm\, h_0'(k), \tag{60}$$

and at the stationary point

$$\phi''(k) = -h''(k)t = \mp h_0''(k)t;\ \phi = (kg-h)t = \pm\,(kh_0' - h_0)t. \tag{61}$$

Therefore (II.12) yields, for $t > 0$

$$u \sim \sum_{\pm}\left(\frac{2\pi}{\lambda}\right)^{1/2} a_{\pm}(k)e^{\mp i\frac{\pi}{4}}\,|th_0''(k)|^{-1/2}\exp\left\{i\lambda\left[\pm(kh_0' - h_0)t\right]\right\}; \qquad x = \pm\, h_0'(k)t. \tag{62}$$

(Here we have used the fact that, $h_0''(k) > 0$, hence sig ϕ'' = sgn $\phi'' = \mp 1$ for $t > 0$.)

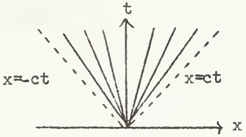
We now note that (57) and (62) agree exactly, provided we set

$$s_{0\pm} = 0; \quad \tilde{z}_{\pm}(k) = \left(\frac{2\pi}{\lambda}\right)^{1/2}\frac{a_{\pm}(k)}{c}\ e^{\mp i\pi/4}\ . \tag{63}$$

Here $a_{\pm}(k)$ are the fourier transforms of the initial data, given by (58). Thus we have verified our general method, and have determined $s_{0\pm}$ and $\tilde{z}_{\pm}$.

The asymptotic solution of the initial value problem for the case of variable coefficients $c(x)$, $b(x)$ can now be completed by the method of section (2.7) with s_0 and $\tilde{z}_{\pm}(p)$ given by (63). Of course, it is clear that in that case we should replace c and b in (63) by $c(0)$ and $b(0)$.* The details of the construction will be omitted.

For the case of constant coefficients, the rays assosciated with each wave are illustrated in the following figure.



From (12) we see that for $-\infty < k < \infty$, $-c < h_0'(k) < c$ hence the rays all lie within the characteristic cone $x = \pm ct$ emanating from the origin. The origin itself is a caustic of the ray family and the asymptotic solution is singular there. From (12) we note that $[h_0''(k)]^{-1/2} = O(|k|^{3/2})$ as $k \to \pm \infty$, therefore the asymptotic solution also becomes singular on the characteristic cone unless $a_{\pm}(k) = O(|k|^{-3/2})$. Singularities of asymptotic solutions on caustics and other special manifolds (such as shadow boundaries) are well known phenomena in the asymptotic theory of partial differential equations. The singularity on the characteristic cone appears to be another phenomenon of this type.

For the case of variable coefficients the ray picture will of course be similar to the figure except that the rays will, in general, be curves lying within the characteristic conoid.

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*(63) depends on b as well as c because $h_0(k) = \sqrt{c^2k^2+b^2}$ appears in (58).

As a special case of our method we obtain the asymptotic expansion of the Riemann function for (1) which is a solution of the initial value problem with

$$u(0,x) = 0 \quad , u_t(0,x) = \delta(x). \tag{64}$$

Since $\delta(x) = \lambda\delta(\lambda x)$, we may take $u_o(x') = 0$ and $u_1(x') = \delta(x')$ in (50). Then (58) yields

$$a_{\pm}(k) = \pm i[4\pi h_o(k)]^{-1} \tag{65}$$

and (63) becomes

$$s_o = 0, \quad \tilde{z}_{\pm}(k) = \left[2(2\pi\lambda)^{1/2} c\, h_o(k)\right]^{-1} e^{\pm i\pi/4}. \tag{66}$$

3.5 Slowly varying initial data

In view of our definetion (50) of "rapidly varying" initial data, it is appropriate to refer to initial data of the form

$$u(0,x) = u_o(x;\lambda) = u_o(x), \; u_t(0,x) = u_1(x;\lambda) = \lambda u_1(x) \tag{67}$$

as slowly varying. In terms of the transformation (1.31) the initial value problem (1,67) is equivalent to the initial value problem for (51) with initial data

$$u'(0,x') = u_o(x'/\lambda), \; u'_{t}{}'(0,x') = u_1(x'/\lambda). \tag{68}$$

The asymptotic solution of the problem (1,67) is easily obtained by observing that (67) is a special case of oscillatory initial data (20), with $s_1 = s_2 = 0$, $y_1 = u_o$, and $iy_2 = u_1$. As we saw in section (3.1) the asymptotic solution of such problems consists of a sum of waves, each of which is generated by data of the form (3). If we set $s_o(x) = 0$ in (3) it becomes

$$u_o(x;\lambda) \sim z_o(x), \quad u_1(x,\lambda) \sim -i\lambda h(0) z_o(x); \; h(0) = \pm |b(x)|. \qquad (69)$$

For the case of constant coefficients, the asymptotic solution can be obtained from (11) by setting $s_o = 0$. This yields

$$u_{\pm}(t,x) \sim z_o(x) \exp\{\mp i\lambda |b| t\}. \qquad (70)$$

The rays, in this case are straight lines parallel to the t-axis, i.e. the group-speed is zero. If we follow the procedure outlined in (15-20) we obtain the asymptotic solution of the initial value problem (1,67) for constant coefficients;

$$u(t,x) \sim u_o(x) \cos(\lambda |b| t) + \frac{u_1(x)}{|b|} \sin(\lambda |b| t). \qquad (71)$$

3.6 discontinuous initial data

It is a characteristic feature of asymptotic solutions of differential equations that discontinuities of the data give rise to additional terms in the asymptotic expansion. In order to illustrate this phenomena, let us suppose that the function $z_o(x)$ in the initial value problem (1,2,3) is smooth, except for a jump discontinuity,

$$[z_0] = z_0(x_0+0) - z_0(x_0-0) \tag{72}$$

at $x = x_0$. For definiteness we shall take $h = \pm h_0$ in (3).

We assume that, in addition to the wave (11) generated by the initial data, two secondary waves $\hat{u}_\pm \sim z_\pm e^{i\lambda s_\pm}$ emanate from the point of discontinuity $(0, x_0)$ on the initial line $t = 0$ The functions $z_\pm$ and $s_\pm$ are given by the method of section 2.7 except that the numbers $s_{0\pm}$ and the functions $\tilde{z}_\pm(p)$ are undetermined. In order to determine them we proceed as in section 3.4. The first step is to complete the construction for the case of constant coefficients.

From (2.47) the rays are given by

$$x = x_0 + gt; \quad g = h'(k) = \pm h_0'(k). \tag{73}$$

From (2.48) and (2.50), $s_\pm$ and $z_\pm$ are given along the rays by

$$s_\pm(t) = s_{0\pm} + \ell t; \ \ell = kh' - h = \pm(kh_0' - h_0) \tag{74}$$

and

$$z_\pm(t) = \tilde{z}_\pm(k) c |t h_0''(k)|^{-1/2}. \tag{75}$$

Thus

$$\begin{gathered} \hat{u}_\pm \sim c\tilde{z}_\pm(k)|h_0''(k)t|^{-1/2} \exp\left\{ i\lambda\left[s_{0\pm} \pm (kh_0' - h_0)t\right]\right\} ; \\ x = x_0 \pm h_0'(k)t. \end{gathered} \tag{76}$$

Let us now determined $s_{0\pm}$ and $\tilde{z}_{\pm}$ by asymptotic expansion of the exact solution, which is given by (21). The leading term of the asymptotic expansion of the double integral (21) comes from the neighborhood of the stationary point and has already been computed (see (28, 33).) It can be shown that the next term in the asymptotic expansion comes from the neighborhood of the discontinuity line $\sigma = x_0$ of the integrand and can be determined as follows:

We first divide the integral (21) into two parts, w_2 corresponding to the strip $x_0-\epsilon \leq \sigma \leq x_0 + \epsilon$, and w_1 corresponding to the complementary region of the σ, k-plane. We choose ϵ small enough so that the stationary point lies in the domain of w_1. w_2 may be written in the form

$$w_2 = w_+ + w_- \tag{77}$$

where

$$w_{\pm} = \int_{-\infty}^{\infty} v_{\pm}(k) e^{i\lambda\left[kx\mp h_0(k)t\right]}\, dk. \tag{78}$$

and

$$\begin{aligned} v_{\pm}(k) &= \frac{\lambda}{4\pi}\int_{x_0-\epsilon}^{x_0+\epsilon} q_{\pm}(\sigma,k) z_0(\sigma) e^{i\lambda\left[s_0(\sigma)-k\sigma\right]}\, d\sigma; \\ q_{\pm}(\sigma,k) &= 1 \pm \frac{h_0\left[s_0'(\sigma)\right]}{h_0(k)} . \end{aligned} \tag{79}$$

Next we divide each integral (79) into two parts corresponding to the intervals $x_0- \epsilon \leq \sigma \leq x_0$ and $x_0 \leq \sigma \leq x_0 + \epsilon$, and integrate each by parts. It is easily shown (by a second integration by parts) that the resulting integrals are of order $1/\lambda$ while the boundary termsare of order 1. Of the four boundary terms, the two which come from the points $x_0 - \epsilon$ and $x_0 + \epsilon$ must be ignored for they are exactly cancelled by boundary contributions from w_1. Then the boundary terms corresponding to the points $x_0 - 0$ and $x_0 + 0$ yield

$$v_{\pm}(k) \sim \frac{-[z_0]}{4\pi i} \frac{q_{\pm}(x_0,k)}{s_0'(x_0)-k} e^{i\lambda[s_0(x_0)-kx_0]} \tag{80}$$

We now insert (80) in (78). This yields

$$w_{\pm} \sim \frac{-[z_0]}{4\pi i} e^{i\lambda s_0(x_0)} \int_{-\infty}^{\infty} \frac{q_{\pm}(x_0,k)}{s_0'(x_0)-k} e^{i\lambda\phi_{\pm}(k)} dk, \tag{81}$$

where

$$\phi_{\pm}(k) = k(x-x_0) \pm h_0(k)t. \tag{82}$$

Finally we apply the stationary phase formula to (81). The stationary point is determined by

$$\phi'(k) = x-x_0 \pm h_0'(k)t = 0, \tag{83}$$

and at that point,

$$\phi'' = \mp h_0''(k)t, \quad \phi = \pm\left[kh_0' - h_0\right]t. \tag{84}$$

Then, for $t > 0$, (II.12) yields $w_{\pm} \sim \hat{u}_{\pm}$ where

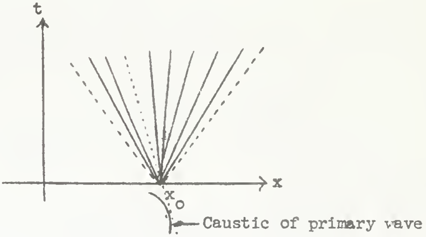
$$\hat{u}_{\pm} = \frac{i[z_0]e^{\pm i\ \pi/4}}{2(2\pi\lambda)^{1/2}} \ \frac{\{1 \pm h_0[s_0'(x_0)]/h_0(k)\}}{s_0'\ (x_0)-k} |h_0''(k)t|^{-1/2} \exp\left\{i\lambda\left[s_0(x_0) \pm (kh_0' - h_0)t\right]\right\} ,$$

$$x = x_0 \pm h_0'(k)t. \tag{85}$$

Since the integral w_1 contains the stationary point, its asymptotic expansion is again given by (28,33), and corresponds to the primary wave. Therefore the two secondary waves correspond to (85). If we now compare (85) and (76) we see that they agree exactly provided we set

$$s_{0\pm} = s_0(x_0), \quad \tilde{z}_{\pm}(k) = \frac{i[z_0]e^{\pm i\pi/4}}{2(2\pi\lambda)^{1/2}c} \quad \frac{\{1 \pm h_0[s_0'(x_0)]/h_0(k)\}}{s_0'(x_0)-k} . \tag{86}$$

We note that the primary wave (28) is of order 1 while the secondary waves are of order $\lambda^{-1/2}$. The rays of the primary wave are illustrated in the figure of section 3.1. Those of the secondary waves are illustrated in the following figure:



They all lie within the characteristic cone (dashed lines) emanating from the point $(0,x_0)$. One ray (dotted line) coincides with a ray of the primary wave. From (7) we see that the eqution of this ray is

$$x = x_0 + h_0'[s_0'(x_0)]t. \tag{87}$$

From (73) we see that the ray

$$x = x_0 + h_0'(k)t \tag{88}$$

of the secondary wave $\hat{u}_+$ coincides with it, provided

$$k = s_0'(x_0). \tag{89}$$

But for this value of k we see from (86) that $\tilde{z}_+(k)$ is infinite (while $\tilde{z}_-$ is indeterminate). Thus the secondary wave $\hat{u}_+$ is infinite along the ray which co-incides with a ray of the primary wave. Since $[h_0''(k)]^{-1/2} = O(|k|^{3/2})$ as $k \to \infty$, it is easily seen that both secondary waves become singular on the characteristic cone.

As in section (3.4), the formulas (86) for $s_0 \pm$ and $\tilde{z}_\pm(p)$ can be used to complete the construction of the secondary waves $\hat{u}_\pm$ for the case of variable coefficients. In doing so we must take $c = c(x_0)$ and $b = b(x_0)$ in (86). Once $\hat{u}_\pm$ are determined, the asymptotic solution of the initial value problem (1,2,3) (with z_0 discontinuous at $x = x_0$ and $h = +h_0$) is given by

$$u = u_+ + \hat{u}_+ + \hat{u}_- + O(1/\lambda). \tag{90}$$

Here u_+ is given by (11) for the case of constant coefficients. For the case of variable coefficients $\hat{u}_\pm$ can be determined by the method of section 2, using (86).

4. Initial-boundary value problems for the inhomogeneous equation

In this section we shall develop methods for obtaining the asymptotic solution of initial-boundary value problems for the inhomogeneous equation

$$c^2(x)u_{xx}-u_{tt}-\lambda^2b^2(x)u = \lambda^2f(t,x;\lambda). \tag{1}$$

Without loss of generality, we may assume that the initial data, which we prescribe at $t = t_1 < 0$, is homogeneous, i.e.

$$u(t_1,x) = u_t(t_1,x) = 0. \tag{2}$$

(The solution of the problem for (1) with inhomogeneous initial data is then given by $u_1 + u_2$ where u_1 is the solution of (1,2) and u_2 is a solution of the problem (3.1,2) for the homogeneous equation.).

We shall assume that the source term λ^2f gives rise to one or more waves. The sum of these waves provides the asymptotic solution of the initial value problem. For initial-boundary value problems we assume that a secondary reflected wave is produced (as in section 3.3) when a primary wave is incident on the boundary. Since we have already examined this phenomenon, at least in a simple case, we shall restrict our attention here to problems without boundaries.

In the following subsections we shall construct the waves produced by several different types of sources.

4.1 rapidly varying source

The source-function will be said to be rapidly-varying if it is of the form

$$\lambda^2f(t,x;\lambda) = \lambda^2g(\lambda t,\lambda x). \tag{3}$$

In terms of the transformation (1.16) the initial-value problem (1,2,3) is then equivalent to the problem

$$(c')^2 u'_{x'x'} - u'_{t't'} - (b')^2 u' = g(t',x'), \tag{4}$$

$$u'(\lambda t_1, x') = u'_{t'}(\lambda t_1, x') = 0. \tag{5}$$

Hence we see that this type of source is quite natural for the problem without a parameter in the differential equation and with homogeneous initial conditions in the "distant past", i.e. at $t' = \lambda t_1$ $(t_1 < 0,\ \lambda \to \infty)$.

Let us assume that the function $g(t',x')$ has compact support.* Then for $\lambda \to \infty$ the support of $g(\lambda t, \lambda x)$ shrinks to the point $(t,x) = (0,0)$. Therefore we assume that the source (3) gives rise to waves emanating from this point. If we set

$$u = u_+ + u_- \ ; \quad u_\pm \sim z_\pm e^{i\lambda s_\pm}, \tag{6}$$

then the functions $z_\pm$ and $s_\pm$ are given by the method of section (2.7) except that the numbers $s_{o\pm}$ and the functions $\tilde{z}_\pm(p)$ are undetermined. As in section (3.4), we shall carry out the construction for the case of constant coefficients, and then shall determine $s_{o\pm}$ and $\tilde{z}_\pm$ by comparison with the asymptotic expansion of the exact solution.

The asymptotic solution for the case of constant coefficients is the same as that of section (3.4) except for the values of $s_{o\pm}$ and $\tilde{z}_\pm$. Therefore $u_\pm$ is given along the rays

$$x = \pm h_o'(k)t \tag{7}$$

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* Actually it is sufficient to assume that the decay at infinity of $g(t',x')$ is sufficiently rapid for the integral (11) to exist.

by

$$u_{\pm} = c\tilde{z}_{\pm}(k)|th_0''(k)|^{-1/2} \exp\left\{ i\lambda\left[s_{0}\pm(kh_0' - h_0)t\right]\right\}. \tag{8}$$

The exact solution of the initial value problem (1,2) is given by (I.23) of appendix I. We insert (3) in that equation and then introduce the transformation $x' = \lambda y$, $t' = \lambda s$. The result is

$$u(t,x;\lambda) = \sum \frac{1}{4\pi i} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx' \int_{\lambda t_1}^{\lambda t} dt' \, \frac{g(t',x')}{h(k)} \, e^{-i[kx'-ht']+i\lambda[kx-ht]}. \tag{9}$$

Here $h = \pm h_0(k)$. For $\lambda \to \infty$ it is clear from (9) that

$$u \sim \sum \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{a(k,h)}{h} \, e^{i\lambda\phi(k)} \, dk, \tag{10}$$

where

$$a(k,h) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' \, g(t',x')e^{-i[kx'-ht']}, \tag{11}$$

and

$$\phi(k) = kx-ht. \tag{12}$$

We now apply the method of stationary phase (appendix II) to (10). The stationary point is determined by

$$\phi'(k) = x-h't = 0, \tag{13}$$

and at the stationary point,

$$\phi = (kh'-h)t = \pm(kh_0' - h_0)t; \; \phi'' = -h''t = \mp h_0'' \, t. \tag{14}$$

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*Actually it is sufficient to assume that the decay at infinity of $g(t; x')$ is sufficiently rapid for the integral (11) to exist.

According to (3.12) $h_o'' > 0$ therefore for $t > 0$ $\operatorname{sgn} \phi'' = \mp 1$. It follows from (II.12) that $u \sim u_+ + u_-$, where

$$u_\pm \sim \frac{a(k,\pm h_o)}{2(2\pi\lambda)^{1/2} h_o} \, |t h_o''(k)|^{-1/2} \exp\left\{ i\lambda\left[\pm(k h_o' - h_o)t\right] \mp 3i\,\pi/4 \right\};$$

$$x = \pm h_o'(k)t. \tag{15}$$

This equation agrees exactly with (8) if we set

$$s_{o\pm} = 0, \quad \tilde{z}_\pm(k) = \frac{a[k, \pm h_o(k)]}{2(2\pi\lambda)^{1/2} c h_o(k)} \, e^{\mp 3i\pi/4}; \quad h_o = \sqrt{c^2k^2 + b^2}\,. \tag{16}$$

Here $a(k,h)$ is given by (11). (16) can be used to complete the asymptotic solution of the problem with variable coefficient functions. In so doing we must set $c = c(0)$ and $b = b(0)$ in (16).

Two special cases of the results of this section are of some interest. One is the point source with

$$g(t',x') = g(t')\delta(x'). \tag{17}$$

The other is the source

$$g(t',x') = \delta(t')\delta(x') \tag{18}$$

which determines the Green's function of the initial value problem. For the point source, we see from (11) that

$$a(k,h) = a(h) = \int_{-\infty}^{\infty} g(t')e^{iht'}\, dt', \tag{19}$$

while, for the Green's function

$$a(k,h) \equiv 1. \tag{20}$$

4.2 Source with rapid time variation

Let us now consider a source function of the form

$$f(t,x;\lambda) = g\left\{ \lambda[t-t_o(x)],x \right\} . \tag{21}$$

If the function $g(t',x)$ has compact support* then as $\lambda \to \infty$ the support of f shrinks to the curve $t = t_o(x)$. In order to conform to the notation of section 2, we express this curve in the parametric form

$$t = t_o(\sigma) \ , \quad x = x_o(\sigma) \equiv \sigma. \tag{22}$$

To simplify our subsequent work it is convenient to assume that $t_o(\sigma)$ is a monotonic function, and for definiteness we shall take

$$t_o'(\sigma) \geq 0. \tag{23}$$

We assume that two waves are produced by the source (21) and that they emanate from the curve (22). We also assume that on this curve, $s(t,x)$ is constant for each wave, i.e.

$$s\left[t_o(\sigma),\sigma\right] = s_{o\pm} = \text{const}. \tag{24}$$

Differentiation of (24) with respect to σ yields the condition

$$-\omega t_o' + k = 0 \text{ or } \omega/k = 1/t_o' = \frac{dx}{dt}. \tag{25}$$

(The condition (25) may be stated as follows: The "phase speed" ω/k is equal to the speed dx/dt of the source.). By squaring (25) we obtain $k^2 = (t_o')^2h^2(k,\sigma)$, and since h depends on k only as a function of k^2 this equation has two solutions

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* It is sufficient to assume that the integral (44) exists.

$$k = \pm k_o(\sigma). \tag{26}$$

(By using the formula (2.6) for the hamiltonian h it is easy to find the solutions (26) explicitly. However, since we wish to keep our methods as general as possible we shall not make use of the explicit expression for k_o.).

We now impose the condition that the solutions (26) be real, and we take $k_o(\sigma) \geq 0$. For the case of equation (1) it is easy to show that this condition is equivalent to

$$|c(\sigma)t_o'(\sigma)| < 1 \quad \text{or} \quad |c| < |\frac{dx}{dt}|, \tag{27}$$

i.e. the source speed must be greater than the characteristic speed $|c|$. (Geometrically, the condition is that the curve (22) should be space-like.). From (25) and (2.6)

$$\omega = \pm\omega_o(\sigma) = \pm k_o(\sigma)/t_o' = \pm\, h_o[k_o(\sigma),\sigma] \tag{28}$$

and from (2.8)

$$g = g_o(\sigma) = \pm \frac{\partial h_o}{\partial k}(\pm\, k_o,\sigma) = \frac{\partial h_o}{\partial k}(k_o,\sigma). \tag{29}$$

The value of the function z for each wave is determined by its value

$$z[t_o(\sigma),\sigma] = z_\pm(\sigma) \tag{30}$$

on the curve (22). If $z_\pm(\sigma)$ and the constants $s_{o\pm}$ were known, the construction of the two waves could be completed by the methods of section 2. We shall determine $z_\pm$ and $s_{o\pm}$ by completing the construction for the case of constant coefficients and comparing the result with the asymptotic expansion of the exact solution.

If the coefficients are constant, the rays are given by*

$$x = \sigma + g_0(\sigma)\,[t-t_0(\sigma)] = \sigma + h_0'[k_0(\sigma)]\,[t-t_0(\sigma)] \tag{31}$$

and s is given along the rays by

$$s = s_{0\pm} + \ (t-t_0), \tag{32}$$

where

$$\lambda = kg-\omega = \pm\left[k_0(\sigma)h_0'(k_0) - h_0(k_0)\right]. \tag{33}$$

From (2.35) and (30), z is given along the rays by

$$z(t) = z_{\pm}(\sigma)\left[j(t_0)/j(t)\right]^{1/2}, \tag{34}$$

where $j(t,\sigma) = dx/d\sigma$. In order to calculate this derivative, using (31), we shall need the value of $k_0'(\sigma) = dk^0/d\sigma$. This is obtained by differentiating implicitly the equation

$$h_0(k_0)t_0' = k_0 \tag{35}$$

which is a consequence of (25), (26), and (28). The result is

$$k_0'(\sigma) = h_0 t_0''/(1-h_0't_0'). \tag{36}$$

Then

$$j(t,\sigma) = dx/d\sigma = 1-h_0't_0' + (t-t_0)h_0''\,h_0t_0''/(1-h_0't_0'). \tag{37}$$

It follows that

$$j(t_0,\sigma) = 1-h_0't_0', \tag{38}$$

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* We note that $g_0(\sigma)$ is the same for both waves, therefore the rays are the same.

and (34) yields

$$z(t) = z_{\pm}(\sigma) \left[(1-h_o' t_o')^2 + (t-t_o) h_o'' \, h_o t_o''\right]^{-1/2} \tag{39}$$

If we denote the two waves by u_+ and u_-, then the asymptotic solution of the initial value problem (1,2,21) is given by $u = u_+ + u_-$, where

$$u_{\pm} \sim z_{\pm}(\sigma) \left[(1-h_o' t_o')^2 + (t-t_o) h_o'' \, h_o t_o''\right]^{-1/2} \exp\left\{ i\lambda \left[s_o \pm (k_o h_o' - h_o)(t-t_o)\right] \right\} ;$$

$$x = \sigma + h_o'(t-t_o). \tag{40}$$

Here, the argument of h_o, h_o', h_o'' is $k_o(\sigma)$.

The exact solution of the problem is given by (I.23). We insert (21) in that equation, change y to σ, and introduce the transformation $t' = \lambda\left[s-t_o(\sigma)\right]$ in the inner integral. This yields

$$u = \sum \frac{\lambda}{4\pi i} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\sigma \int_{\lambda\left[t_1-t_o(\sigma)\right]}^{\lambda\left[t-t_o(\sigma)\right]} dt' \; \frac{g(t',\sigma)}{h(k)} \exp\left\{ i\lambda\left[k(x-\sigma)-h(t-t_o)\right] + iht' \right\} \tag{41}$$

Here $h = \pm h_o(k)$. We assume that

$$t_1 < t_o(\sigma) < t. \tag{42}$$

Then, for $\lambda \to \infty$,

$$u \sim \sum \frac{\lambda}{4\pi i} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\sigma \; \frac{a(h,\sigma)}{h(k)} \; e^{i\lambda\phi(k,\sigma)}, \tag{43}$$

where

$$a(h,\sigma) = \int_{-\infty}^{\infty} g(t',\sigma) e^{iht'} dt' \tag{44}$$

and

$$\phi(k,\sigma) = k(x-\sigma) - h\left[t-t_o(\sigma)\right]. \tag{45}$$

We now apply the method of stationary phase to (43). The stationary point is determined by the conditions

$$\phi_k = x-\sigma-h'\left[t-t_o(\sigma)\right] = 0, \quad \phi_\sigma = -k + ht_o'(\sigma) = 0. \tag{46}$$

By comparison with (25) we see that (46) implies that

$$k = \pm k_o(\sigma). \tag{47}$$

and

$$x = \sigma + h_o'\left[k_o(\sigma)\right]\left[t-t_o(\sigma)\right]. \tag{48}$$

Here we have used the fact that $\pm h_o'(\pm k_o) = h_o'(k_o)$. (45) and (48) now yield

$$\phi = \pm\left[k_o h_o' - h_o\right]\left[t-t_o\right]. \tag{49}$$

In order to apply (II.12) we require the second derivatives,

$$\phi_{kk} = -h''\left[t-t_o\right], \quad \phi_{k\sigma} = -1 + h't_o', \quad \phi_{\sigma\sigma} = ht_o''. \tag{50}$$

The derivatives are evaluated at the stationary point, $[k,\sigma] = [\pm k_o(\sigma),\sigma]$. We set

$$(\phi_{\nu j}) = \begin{pmatrix} \phi_{kk} & \phi_{k\sigma} \\ \phi_{\sigma k} & \phi_{\sigma\sigma} \end{pmatrix}. \tag{51}$$

Then (50) and (51) yield

$$\det(\phi_{\nu j}) = -\left[(1-h't_o')^2 + (t-t_o)h''\,h\,t_o''\right]$$

$$= -\left[(1-h_o'\,t_o')^2 + (t-t_o)h_o''\,h_o t_o''\right]. \tag{52}$$

Again we have used the fact that $\pm h_o'(\pm k_o) = h_o'(k_o)$.

For $t = t_o$ we note that $\det(\phi_{\nu j}) = -1$. Therefore, in a neighborhood of $t = t_o$ (and in fact until the ray reaches a caustic), $\mathrm{sig}(\phi_{\nu j}) = 0$. Therefore we see from (II.12) that $u \sim u_+ + u_-$, where

$$u_\pm \sim \frac{\pm a(\pm h_o,\sigma)}{2ih_o}\left[(1-h_o't_o')^2 + (t-t_o)h_o''h_o t_o''\right]^{-1/2} \exp\left\{ i\lambda\left[\pm(k_o h_o'-h_o)\ (t-t_o)\ \right]\right\} ;$$

$$x = \sigma + h_o'(t-t_o). \tag{53}$$

We see that (53) and (40) agree exactly if we set

$$s_{o\pm} = 0, \quad z_\pm(\sigma) = \frac{\pm a(\pm\, h_o,\sigma)}{2ih_o} . \tag{54}$$

Here $h_o = h_o[k_o(\sigma)]$, and $a(h,\sigma)$ is given by (44). For the case of variable coefficients (54) supplies the values of $s_{o\pm}$ and $z_\pm(\sigma)$ necessary to complete the construction of the asymptotic solution. In that case we must set $h_o = h_o[k_o(\sigma),\sigma]$ in (54).

An interesting special case of the results of this subsection is a source function of the form

$$f(t,x;\lambda) = \delta\left[\lambda(t-t_o)\right]\gamma(x) = \frac{1}{\lambda}\,\delta\left[t-t_o(x)\right]\gamma(x) . \tag{55}$$

In this case $g(t',\sigma) = \delta(t')\gamma(\sigma)$ and $a(h,\sigma) = \gamma\ (\sigma)$. It follows that

$z_{\pm}(\sigma) = \pm\chi(\sigma)/2ih_o$ and

$$u \sim u_{+} + u_{-} = \frac{\chi(\sigma)}{h_o}\left[(1-h_o't_o')^2 + (t-t_o)h_o''h_ot_o''\right]^{-1/2} \sin\left[\lambda(k_oh_o'-h_o)(t-t_o)\right]; \tag{56}$$

$$x = \sigma + h_o'(t-t_o).$$

If f is of the form (55) where t_o is a constant, then (25) implies that $k_o(\sigma) = 0$. Hence $h_o = |b|$, $x = \sigma$, and (56) becomes

$$u \sim -\frac{\chi(x)}{b} \sin\left[\lambda b(t-t_o)\right]. \tag{57}$$

A source function of the form (21) is said to have rapid time variation. One of the form

$$f(t,x;\lambda) = g\left\{t,\lambda\left[x-x_o(t)\right]\right\} \tag{58}$$

is said to have rapid space variation. Since the treatment of (58) is similar to that of (21) we shall not examine it further. A special case of (58) is the moving point source,

$$f(t,x;\lambda) = \gamma(t)\delta\left[\lambda(x-x_o)\right] = \frac{1}{\lambda}\gamma(t)\delta\left[x-x_o(t)\right]. \tag{59}$$

4.3 Oscillatory point source.

A source function of the form

$$\lambda^2 f(t,x;\lambda) = \lambda^2\delta(\lambda x)p(t)e^{-i\lambda q(t)} = \lambda\delta(x)p(t)e^{-i\lambda q(t)} \tag{60}$$

will be said to represent an oscillatory point source located at the origin.

In constructing the asymptotic solution of (1, 2, 60) it is natural to assume that waves emanate from the t-axis: $x = 0$, $t = \tau$ $(t_1 \leq \tau)$ with a phase given at $x = 0$ by

$$s(\tau,0) = s_o(\tau) = -q(\tau). \tag{61}$$

Differentiation of (61) with respect to τ leads to

$$\omega = h(k) = \pm h_o(k) = q'(\tau); \qquad h_o = \sqrt{c^2k^2 + b^2}. \tag{62}$$

This equation has real solutions k if and only if $[q'(\tau)]^2 > b^2(0)$. For definiteness we shall assume that*

$$q'(\tau) > b(0). \tag{63}$$

If (63) is satisfied then (62) has real solutions k only for

$$h = + h_o, \tag{64}$$

and then

$$k = \pm k_o(\tau). \tag{65}$$

As in the preceeding section, the explicit formula for $k_o(\tau)$ is easily obtained but will not be used. The corresponding values of g at the source are clearly given by

$$g = h_o' (\pm k_o) = \pm h_o'(k_o). \tag{66}$$

The value of the function z is determined by its value

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*In terms of the physical interpretation of (1) given in the introduction, the condition (63) is that the frequency of the source should be greater than the plasma frequency ϕ. It is well-known that waves of frequency less than ϕ are exponentially damped in a plasma.

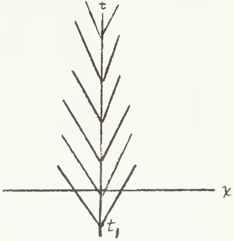
$$z(\tau,0) = z_{\pm}(\tau)$$

at the source. If $z_{\pm}(\tau)$ were known the solution could be completed by the methods of section (2). As usual, we shall determined the value of $z_{\pm}(\tau)$ by compliting the construction for the case of constant coefficients and by comparison with the expansion of the exact solution.

If the coefficients are constant, the rays are given by

$$x = \pm h_o'[k_o(\tau)](t-\tau), \qquad t_1 \leq \tau \leq t; \tag{67}$$

and are illustrated in the following figure.



Those corresponding to the upper (lower) sign in (67) proceed to the right (left) with increasing t. As usual, s is given along each ray by

$$s = s_o(\tau) + \ell(t-\tau) = -q(\tau) + (t-\tau)\left[k_o(\tau)h_o'(k_o) - q'(\tau)\right]. \tag{68}$$

(Here we have made use of the equations $\pm h_o'(\pm k_o) = h_o'(k_o)$ and $h_o(k_o) = q'(\tau)$.) The function z is given along each ray by

$$z(t) = z_{\pm}(\tau)\left[j(\tau)/j(t)\right]^{1/2} \tag{69}$$

where $j(t) = j(t,\tau) = dx/d\sigma$. In order to calculate j, using (67) we first obtain the value of $k_o'(\tau)$. This is easily obtained by implicit differentiation of (62) with $h = \pm h_o$. The result is

$$k_o'(\tau) = q''(\tau)/h_o'(k_o). \tag{70}$$

Then

$$j(t,\tau) = \pm[h_o'(k_o)]^{-1}\{h_o''(k_o)q''(\tau)(t-\tau)-[h_o'(k_o)]^2\} \tag{71}$$

and

$$j(\tau,\tau) = +h_o'(k_o). \tag{72}$$

(69) now yields

$$z(t) = z_{\pm}(\tau)\left[1 - \frac{(t-\tau)q''(\tau)h_o''(k_o)}{[h_o'(k_o)]^2}\right]^{-1/2}, \tag{73}$$

hence the asymptotic solution of the intial-value problem (1, 2, 60) is given parametrically in terms of τ by

$$u \sim z_{\pm}(\tau)\left[1 - \frac{(t-\tau)q''(\tau)h_o''(k_o)}{[h_o'(k_o)]^2}\right]^{-1/2} \exp\left[i\lambda\{-q(\tau)+(t-\tau)[k_o(\tau)h_o'(k_o)-q'(\tau)]\}\right];$$

$$x = \pm\, h_o'[k_o(\tau)](t-\tau); \qquad t_1 \leq \tau \leq t. \tag{74}$$

The exact solution of the problem is given by (I.23). We set

$$\hat{p}(\tau) = \left\{\begin{matrix} p(\tau), & t_1 \leq \tau \leq t \\ 0\quad, & \text{otherwise} \end{matrix}\right\}. \tag{75}$$

Then if we insert (60) into (I.23) we obtain

$$u(t,x) = \frac{\lambda}{4\pi i}\sum_{h=\pm h_o}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\tau\, \frac{\hat{p}(\tau)}{h(k)}\, e^{i\lambda\phi(k,\tau)}. \tag{76}$$

Here

$$\phi(k,\tau) = -q(\tau) + kx - (t-\tau)h(k). \tag{77}$$

We now apply the method of stationary phase to (76). Stationary points are determined by the conditions

$$\phi_k = x - (t-\tau)h'(k) = 0; \qquad \phi_\tau = -q'(\tau) + h(k) = 0. \tag{78}$$

Again assuming that (63) is satisfied, we see that (78) is satisfied only for $h = +h_o$ and then $k = \pm k_o(\tau)$. Thus only one term of (76) yields a stationary point, and at this point

$$x = (t-\tau)h_o'[\pm k_o(\tau)] = \pm (t-\tau)h_o'[k_o(\tau)]. \tag{79}$$

It is now an easy matter to compute the matrix

$$(\phi_{\nu_j}) = \begin{pmatrix} \phi_{kk} & \phi_{k\tau} \\ \phi_{\tau k} & \phi_{\tau\tau} \end{pmatrix} \tag{80}$$

at the stationary point. Then

$$\det(\phi_{\nu j}) = (t-\tau)q''(\tau)h_o''[\pm k_o] - [h_o'(\pm k_o)]^2 = (t-\tau)q''(\tau)h_o''(k_o) - [h_o'(k_o)]^2. \tag{81}$$

In a neighborhood of $t = \tau$ the determination is negative, hence $\text{sig}(\phi_{\nu j}) = 0$. The stationary phase formula, (II.12) now yields

$$u \sim \frac{1}{2i} \{[h_o'(k_o)]^2 - (t-\tau)q''(\tau)h_o''[k_o]\}^{-1/2} \frac{p(\tau)}{q\ (\tau)} \exp\left[i\lambda\{-q(\tau)+(t-\tau)[k_o(\tau)h_o'(k_o)-q'(\tau)]\}\right];$$

$$x = \pm h_o'[k_o(\tau)](t-\tau); \qquad t_i \leq \tau \leq t. \tag{89}$$

Comparison of (74) and (89) shows that they agree exactly if we set

$$z_\pm(\tau) = \frac{p(\tau)}{2ih_o'[k_o(\tau)]q'(\tau)} . \tag{90}$$

For the case of variable coefficients, (90) supplies the formula for $z_{\pm}(\tau)$ necessary to complete the construction of the asymptotic solution. Important special cases of the results of this section are obtained by setting

$$q(t) = \nu t \qquad [\nu > b(0)] \tag{91}$$

and

$$q(t) = \nu t \quad [\nu > b(0)]; \qquad p(t) \equiv p_0. \tag{92}$$

(92) yields a time-harmonic point source of frequency $\lambda\nu$, and (91) may be interpreted as an amplitude-modulated time-harmonic source.

4.4 Slowly varying source

As a final example of our method of solving the initial value problem (1,2) we consider the slowly varying source function

$$\lambda^2 f(t,x;\ \lambda) = g(t,x). \tag{93}$$

We assume that $g(\tau,x) \equiv 0$ for $t_2 < \tau$ where t_2 is a constant, and we examine the solution $u(t,x)$ for $t_2 < t$. By virture of (2) we may define $g(\tau,x) \equiv 0$ for $\tau < t_1$. Thus

$$g(t,x) \equiv 0 \text{ for } t < t_1 \text{ or } t_2 < t. \tag{94}$$

Then, in general, $g(t,x)$ will have a discontinuity on the line $t = t_1$, and perhaps on other curves of the form

$$t = t_0(\sigma), \quad x = \sigma. \tag{95}$$

As in section (4.2) we take $t_0'(\sigma) \geq 0$, for simplicity.

We assume that each discontinuity curve (95) produces two waves and that on the curve s is constant for each wave. We shall calculate the waves produced by one discontinuity curve (95). The calculation is identical to that of section 4.2 and we obtain the two waves $u_\pm$ given by (40). Again we must impose the condition (27) and $z_\pm(\sigma)$, $s_{0\pm}$ are undetermined. In order to determine them we use the exact solution (I.23). By virtue of (94) we may write it in the form

$$u = \sum \frac{1}{4\pi i} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau \ \frac{g(\tau,\sigma)}{h} e^{i\lambda[k(x-\sigma)-(t-\tau)h]}; \ h = \pm h_0(k). \tag{96}$$

Now let

$$\gamma(\sigma) = [g] = g\left[t_o(\sigma)+0,\ \sigma\right]-g\left[t_o(\sigma)-0,\sigma\right] \tag{97}$$

be the magnitude of the jump discontinuity of g. Then by integration by parts

$$\int_{-\infty}^{\infty} \frac{g(\tau,\sigma)}{i\ \lambda\ h}\, e^{i\lambda\tau h}\, d\tau \sim - \gamma(\sigma)e^{i\lambda t_o(\sigma)h} . \tag{98}$$

Next we insert (98) in (96). The result is

$$u \sim \sum \quad - \frac{\lambda}{4\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\sigma \quad \gamma(\sigma)e^{i\lambda\phi(k,\sigma)} \tag{99}$$

where

$$\phi(k,\sigma) = k(x-\sigma) - h\left[t-t_o(\sigma)\right]. \tag{100}$$

The integral (99) may be evaluated asymptotically by the method of stationary phase. Since (99) is of the same from as (43), we may obtain the result at once by replacing $\frac{a(h,\sigma)}{ih} = \frac{\pm a(\pm h_o,\sigma)}{ih_o}$ by $-\gamma(\sigma)$ in (53). We may then determine $c_{o\pm}$ and $z_{\pm}(\sigma)$ by comparison with (40). In this way we obtain

$$s_{o\pm} = 0, \quad z_{\pm}(\sigma) = -\frac{\gamma(\sigma)}{2} = -\frac{1}{2}[g], \tag{101}$$

and these results may be used to obtain the asymptotic solution of the initial value problem with variable coefficients.

Appendix I. Exact solution of the initial value problem for constant coefficients

We first consider the initial value problem for the homogeneous equation,

$$c^2 u_{xx} - u_{tt} - \lambda^2 b^2 u = 0, \tag{1}$$

$$u(0,x) = u_o(x;\lambda), \quad u_t(0,x) = u_1(x;\lambda). \tag{2}$$

We consider a solution of (1) in the form of a superposition of plane waves,

$$u(t,x) = \sum_{\pm} \int_{-\infty}^{\infty} a_{\pm}(k,\lambda) e^{i\lambda(kx \pm h_o t)} dk; \quad h_o = \sqrt{c^2k^2+b^2}. \tag{3}$$

In order to satisfy the initial conditions (2) we set

$$u(0,x) = \sum \int a_{\pm} e^{i\lambda kx}\, dk = u_o(x;\lambda), \tag{4}$$

$$u_t(0,x) = \sum \int \mp i\lambda h_o a_{\pm} e^{i\lambda kx}\, dk = u_1(x;\lambda). \tag{5}$$

Then, by means of the fourier transform formulas,

$$g(x,\lambda) = \int a(k,\lambda) e^{i\lambda kx}\, dk; \quad a(k,\lambda) = \frac{\lambda}{2\pi} \int g(x,\lambda) e^{-i\lambda kx}\, dx, \tag{6}$$

we may invert (4) and (5). This yields

$$a_+ + a_- = \frac{\lambda}{2\pi} \int u_o(x;\lambda) e^{-i\lambda kx}\, dx, \tag{7}$$

$$a_+ - a_- = \frac{i}{2\pi h_o} \int u_1(x;\lambda) e^{-i\lambda kx} dx, \tag{8}$$

and therefore

$$a_{\pm}(k,\lambda) = \frac{1}{4\pi} \int \left[\lambda u_o(x;\lambda) \pm \frac{i}{h_o} u_1(x;\lambda)\right] e^{-i\lambda kx}\, dx. \tag{9}$$

Finally the solution of (1,2) is obtained by inserting (9) in (3).

This yields

$$u(t,x) = \frac{1}{4\pi} \sum_{\pm} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\sigma \left[\lambda u_o(\sigma;\lambda) \pm \frac{i}{h_{o(k)}} u_1(\sigma;\lambda)\right] e^{i\lambda\left[k(x-\sigma) \pm h_o t\right]} \tag{10}$$

The solution of the initial value problem for the inhomogeneous equation can be reduced to the solution of (1,2) and the solution of the problem for the inhomogeneous equation with zero initial data. Therefore we consider the problem,

$$c^2 u_{xx} - u_{tt} - \lambda^2 b^2 u = \lambda^2 f(t,x;\lambda) \tag{11}$$

$$u(t_1,x) = u_t(t_1,x) = 0. \tag{12}$$

Here t_1 is a constant. In order to solve (11,12) we set

$$u(t,x) = \int a(t,k) e^{i\lambda kx}\, dk; \; f(t,x;\lambda) = \int \phi(t,k;\lambda) e^{i\lambda kx}\, dk. \tag{13}$$

Then (11) yields

$$a_{tt} + \lambda^2 h_o^2 a = -\lambda^2 \phi, \tag{14}$$

and from (12),

$$a(t_1,k) = a_t(t_1,k) = 0. \tag{15}$$

To solve the initial value problem (14,15) for the inhomogeneous linear ordinary differential equation (14) we apply the method of "variation of parameters" to the two linearly independent solutions $\exp\left\{ \pm i\lambda h_o t \right\}$ of the homogeneous equation; i.e. we set

$$a = v_1 e^{i\lambda h_o t} + v_2 e^{-i\lambda h_o t} \tag{16}$$

and insert (16) in (14). In so doing we require that

$$v_{1t} e^{i\lambda h_o t} + v_{2t} e^{-i\lambda h_o t} = 0 \tag{17}$$

and we obtain

$$i\lambda h_o v_{1t} e^{i\lambda h_o t} - i\lambda h_o v_{2t} e^{-i\lambda h_o t} = -\lambda^2 \phi. \tag{18}$$

(17) and (18) may now be solved for v_{1t} and v_{2t} and then integrated to obtain

$$v_1 = -\frac{\lambda^2}{2i\lambda h_o} \int_{t_1}^{t} e^{-i\lambda h_o s} \phi(s)ds; \quad v_2 = \frac{\lambda^2}{2i\lambda h_o} \int_{t_1}^{t} e^{i\lambda h_o s} \phi(s)ds. \tag{19}$$

We now insert (19) in (16). This yields

$$a = \sum \pm \frac{\lambda}{2ih_o} \int_{t_1}^{t} e^{\pm i\lambda h_o (t-s)} \phi(s,k;\lambda)ds, \tag{20}$$

and it is easily seen that (15) is satisfied. By inverting the fourier transform (13) we see that

$$\phi(s,k;\lambda) = \frac{\lambda}{2\pi} \int f(s,y;\lambda) e^{-i\lambda ky} \, dy. \tag{21}$$

Then

$$a = \sum \pm \frac{i\lambda^2}{4\pi h_o} \int_{-\infty}^{\infty} dy \int_{t_1}^{t} ds \; e^{i\lambda\left[-ky \pm (t-s)h_o\right]} f(s,y;\lambda) \tag{22}$$

and by inserting (22) in (13) we obtain the solution of (11,12),

$$u(t,x;\lambda) = \sum \frac{\lambda^2}{4\pi i} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy \int_{t_1}^{t} ds \, \frac{f(s,y;\lambda)}{h} \, e^{i\lambda\left[k(x-y)-(t-s)h\right]} \,. \qquad (23)$$

Here $h(k) = \pm h_0(k)$.

Appendix II. The method of stationary phase in n dimensions

In this appendix we derive a simple formula for the leading term of the asymptotic expansion (for $\lambda \to \infty$) of a function $f(\lambda)$ defined by a certain type of n-dimensional integral. The method we use is the method of stationary phase. Although numerous references for this method exist, the n-dimensional formula we require is not readily available. We will derive it by a method which is, if not entirely rigorous, relatively simple. Let

$$f(\lambda) = \int_D g(t_1,\dots,t_n)e^{i\lambda\phi(t_1,\dots,t_n)}dt_1\dots dt_n = \int_D g(T)e^{i\lambda\phi(T)}dT \, . \tag{1}$$

A stationary point is a point $T = S = (s_1,\dots,s_n)$ for which the real-valued phase function $\phi(T)$ is stationary, i.e. $\phi_\nu(S) = \phi_{t_\nu}(S) = 0$; $\nu = 1,\dots,n$. We assume that the domain of integration D in euclidean n-space E^n contains exactly one stationary point, and that this point lies in the interior of D. (If D contains several stationary points it can be subdivided into sub-domains each containing one.). A Taylor expansion of $\phi(T)$ about the stationary point then yields

$$\phi(T) = \phi(S) + \frac{1}{2}(t_\nu - s_\nu)(t_j - s_j)\phi_{\nu j} + O\left[|T - S|^3\right]. \tag{2}$$

Here $\phi_{\nu j} = \phi_{t_\nu t_j}(S)$, and the summation convention with respect to repeated indices from 1 to n is employed.

We now introduce a translation and rotation of T-space so that the point $T = S$ is the origin and the matrix $(\phi_{\nu j})$ is diagonal. Then

$$\phi_{\nu j} = r_\nu \, \delta_{\nu j}. \tag{3}$$

Here $r_1, \ldots r_n$ are the (real) eigenvalues of $(\phi_{\nu j})$. It is assumed that $(\phi_{\nu j})$ is non-singular. Hence no eigenvalues vanish. We now set $\gamma = \lambda^{1/2}$ and introduce the transformation

$$x_k = \gamma\, t_k \tag{4}$$

Then

$$\lambda(t_\nu - s_\nu)(t_j - s_j)\phi_{\nu j} = \gamma^2 t_\nu t_j r_\nu \delta_{\nu j} = x_\nu x_j r_\nu \delta_{\nu j} = x_\nu^2 r_\nu \tag{5}$$

and from (2)

$$\exp\left\{i\lambda\phi(T)\right\} = \exp\left\{i\lambda\phi(S) + \tfrac{i}{2} r_k x_k^2 + O(\lambda^{-1/2})\right\} = \exp\left\{i\lambda\phi(s) + \tfrac{i}{2} r_k x_k^2\right\}\left[1 + O(\lambda^{-1/2})\right] \tag{6}$$

Furthermore g(T) has the expansion

$$g(T) = g(S) + O(\lambda^{-1/2}) = g(s)\left[1 + O(\lambda^{-1/2})\right] \tag{7}$$

and

$$\frac{dX}{dT} = \gamma^n = \lambda^{n/2} \tag{8}$$

We now insert (6), (7) and (8) in (1). This yields

$$f(\lambda) \sim g(S)e^{i\lambda\phi(S)} f' \; ; \quad f' = \lambda^{-n/2} \int_{D'} e^{\frac{i}{2} r_k x_k^2}\, dX. \tag{9}$$

Here D' is the image of D under (4). We assume that a negligible error is introduced if we replace D' by the infinite X-space. Then

$$f' \sim \lambda^{-n/2} \prod_{k=1}^{n} \int_{-\infty}^{\infty} e^{(i/2) r_k z^2}\, dz. \tag{10}$$

We now apply to (10) the standard integral formula

$$\int_{-\infty}^{\infty} e^{i/2\ rz^2}\, dz = \left(\frac{2\pi}{|r|}\right)^{1/2} e^{i\pi/4\ \mathrm{sgn}\ r} \quad . \tag{11}$$

Here sgn r = ± 1 if $r \gtrless 0$. Then (9),(10), and (11) yield the required formula

$$f(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} \left[\,|\det(\phi_{\nu j})|\,\right]^{-1/2} g(S) e^{i\lambda\phi(S) + i\pi/4\ \mathrm{sig}(\phi_{\nu j})} ;\ \lambda \to \infty. \tag{12}$$

In (12), sig $(\phi_{\nu j})$ denotes the "signature" of the matrix $(\phi_{\nu j})$ defined by

$$\mathrm{sig}\ (\phi_{\nu j}) = \sum_{k=1}^{n} \mathrm{sgn}\ r_k, \tag{13}$$

where $r_1, \ldots, r_n$ are the eigenvalues of $(\phi_{\nu j})$. We have also used the identity,
$\left|\det(\phi_{\nu j})\right| = \left|\prod_{k=1}^{n} r_k\right| = \prod_{k=1}^{n} |r_k|$.

We note that both $|\det(\phi_{\nu j})|$ and $\mathrm{sig}(\phi_{\nu j})$ are invarient under translation and rotation of the co-ordinate system, hence the result (12) is valid for the original co-ordinates.

References

[1] Keller, J. B. The Geometrical Theory of Diffraction, Proceedings of the Symposium on Microwave Optics, McGill University (1953), or J. Opt. Soc. Amer., 52, 116 (1962).

[2] Lewis, R. M. and Keller, J. B., Asymptotic Methods for Partial Differential Equations: The Reduced Wave Equation and Maxwell's Equations, N.Y.U. Research Report No. EM-194, (1964).

[3] Lewis, R. M., Asymptotic Expansion of Steady-state Solutions of Symmetric Hyperbolic Linear Differential Equations. J. Math. and Mech., 7, 593 (1958).

[4] Lax, P. D., Asymptotic Solutions of Oscillatory Initial Value Problems, Duke Math. J., 24, 627 (1957).

[5] Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. II, Interscience Publishers (1962).

[6] Whitham, G. B., Group vVelocity and Energy Propagation for Three Dimensional Waves, Comm. Pure Appl. Math. 14, 675 (1961).

[7] Budden, K. G., Radio Waves in the Ionosphere, Cambridge Univ. Press (1961).

[8] Landau, L. D. and Lifshitz, E. M., Electrodynamics of Continuous Media, Pergamon Press (1960).

[9] Kay, I. and Keller, J. B., Asymptotic Evaluation of the Field at a Caustic, J. Appl. Phys., 25, 876 (1954).

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